Optimization Qualifying Exam

This is a 90-minute exam. Throughout $\|\cdot\|$ stands for the Euclidean norm.

Question 1 [Cobb-Douglas Utility] Consider a market equilibrium problem. The goal is to distribute each of m goods among n buyers. A quantity $\bar{s}_j > 0$ of each good $j = 1, \ldots, m$ is available. Each buyer $i = 1, \ldots, n$ has a fixed budget $w_i > 0$. Given a price vector $p \in \mathbf{R}^m$, each buyer $i = 1, \ldots, n$ independently determines the quantity x_{ij} of each good $j = 1, \ldots, m$ to purchase by solving the utility maximization problem

$$\begin{array}{ll} \text{maximize} & u_i(x_i) \\ \text{subject to} & p^T x_i \le w_i, \\ \text{variables} & x_i = (x_{i1}, \ldots, x_{im})^T \ge 0 \end{array} \tag{1}$$

where $u_i(\cdot)$ is buyer i's utility function. The solution to this problem is a function of the price vector $p \in \mathbf{R}^m_{++}$. Denote the optimal solution of (1) as $x_i^\star(p)$ for each $i = 1, \ldots, m$ given price $p \in \mathbf{R}^m_{++}$. We call $p^\star \in \mathbf{R}^m_{++}$ a equilibrium price if

$$\sum_{i=1}^n x_i^\star(p) = \bar{s}, \tag{2}$$

where $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_j)^T$. Equation (2) is called the market clearing condition.

In this question, we study an important utility function called the Cobb-Douglas utility:

$$u_i(x_i) = \prod_{j=1}^m x_{ij}^{u_{ij}}, \quad x_{ij} > 0.$$

For simplicity, assume $u_{ij} > 0$ for all i and j, and $\sum_{j=1}^m u_{ij} = 1$ for all i.

(a) (3 points) Rewrite problem (1) as a convex minimization problem. The new problem should have the same optimum x_i as (1). Show that the solution to this problem is unique.

Hint: $\log(x)$ is a strongly concave function. How much of the budget w_i will buyer i spend?

(b) (6 points) Write down the optimality conditions (KKT conditions) of the problem constructed in (a). Write down the dual problem.

(c) (3 points) Find the global solution $x_i^\star$ of the optimization problem (1) using (a) and (b).

Hint: Represent the optimal purchases $x_i^\star$ as an **explicit function** of prices p, utilities $\{u_{ij}\}_{j=1,\ldots,m}$ and budget w_i.

(d) (3 points) Derive the equilibrium price $p^\star$ for the Cobb-Douglas Market.

Hint: Represent the equilibrium price $p^\star$ as an **explicit function** of utilities $\{u_{ij}\}_{i=1,\ldots,n;j=1,\ldots,m}$, budgets $\{w_i\}_{i=1,\ldots,n}$, and supplies $\{\bar{s}_j\}_{j=1,\ldots,m}$.

Question 2 [Convergence of Gradient Descent with different norms] Consider the optimization problem:

$$\text{minimize} \quad f(x).$$

Here $f : \mathbf{R}^n \to \mathbf{R}$ is L-smooth in the ∞-norm with respect to the 1-norm:

$$\|\nabla f(x) - \nabla f(y)\|_\infty \leq L\|x - y\|_1 \quad \text{for all } \ x \in \mathbf{R}^n$$

and μ-strongly convex with respect to the ∞-norm:

$$f(x) \geq f(y) + \langle \nabla f(y), x - y\rangle + \frac{\mu}{2}\|x - y\|_\infty^2 \quad \text{for all } \ x \in \mathbf{R}^n.$$

Further, L and μ satisfy $\mu \leq L$. Let $f_\star$ denote the optimal value of this problem.

You will establish linear convergence of gradient descent (GD) for this problem:

$$x_{k+1} = x_k - \eta \nabla f(x_k).$$

(Hint): Relate the current problem to the one we considered in class when we analyzed GD.

(a) (3 points) Show that the following inequality holds:

$$f(x) \leq f(y) + \langle \nabla f(y), x - y\rangle + \frac{n^{3/2}L}{2}\|y - x\|_2^2 \quad \text{for all } \ x, y \in \mathbf{R}^n.$$

(b) (3 points) Show that f is $\frac{\mu}{n}$-PL with respect to the 2-norm by proving

$$f(x) - f(x_\star) \leq \frac{n}{2\mu}\|\nabla f(x)\|_2^2 \quad \text{for all } \ x, y \in \mathbf{R}^n.$$

(c) (3 points) Show that for appropriate stepsize $\eta > 0$, the iterates produced by GD satisfy

$$f(x_k) - f_\star \leq (1 - h(n, L, \mu))^k \left(f(x_0) - f_\star\right),$$

where h is a function satisfying $h(n, L, \mu) \leq 1$. Give the explicit form of $h(n, L, \mu)$.

(d) (3 points) Find a function $K(n, L, \mu, \epsilon)$ such that for any $\epsilon > 0$, $k \geq K(n, L, \mu, \epsilon)$ ensures

$$f(x_k) - f(x_\star) \leq \epsilon.$$

Give the explicit form of $K(n, L, \mu, \epsilon)$.

(e) (1.5 points) How does the number of iterations required by GD to reach an ϵ-suboptimal solution, in the current setting, compare to the result we proved in class?

(f) (1.5 points) Does the bound on the number of iterations required to achieve an ϵ-accurate solution grow or shrink as the dimension n increases? Do you expect the iterations needed to converge in practice to change in the same way with n, or do you suspect this relation is an artifact of the analysis? Why?

Question 2 Supplementary Material [Proof of Gradient Descent] We shall assume f is L-smooth in the 2-norm with respect to the 2-norm, and μ-PL in the 2-norm.

Proof. By L-smoothness,

$$f(x_k) \leq f(x_{k-1}) - \eta \langle \nabla f(x_{k-1}), x_k - x_{k-1} \rangle + \frac{\eta^2 L}{2} \|x_k - x_{k-1}\|^2.$$

Plugging in the GD update and using $\eta = \frac{1}{L}$, yields

$$f(x_k) \leq f(x_{k-1}) - \frac{1}{2L} \|\nabla f(x_{k-1})\|^2.$$

Now, as f is μ-PL in the 2-norm, we have

$$f(x_{k-1}) - f_\star \leq \frac{\|\nabla f(x_{k-1})\|^2}{2\mu}.$$

Applying the preceding inequality, we reach

$$f(x_k) - f_\star \leq \left(1 - \frac{\mu}{L}\right) (f(x_{k-1}) - f_\star).$$

Recursing, the previous display becomes

$$f(x_k) - f_\star \leq \left(1 - \frac{\mu}{L}\right)^k (f(x_0) - f_\star).$$

Performing some straightforward algebra, we conclude

$$f(x_k) - f_\star \leq \epsilon,$$

whenever $k \geq \frac{L}{\mu} \log \left(\frac{f(x_0) - f_\star}{\epsilon} \right)$. □