CME 307/MS&E 311/OIT 676
Midterm

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Name: ________________________________

SUNetID: ________________________________

(Your email address is yourSUNetID@stanford.edu.)

Problem	**Points Received**	**Points Possible**
1(a)		1
1(b)		1
2(a)		1
2(b)		1
3(a)		1
3(b)		1
Total		**6**

Problem 1. Hotel revenue generation
The Pembrook Inn has 200 identical rooms, which can be booked at three different rates. The manager has provided you with a demand forecast, by room rate and length of stay, for next Monday to Friday (an example of the demand forecasts is given in Table 1). For day $i \in \{\text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}\}$, length of stay $\ell \in \{1, 2, 3\}$, and price $p \in \{p_1, p_2, p_3\}$, the demand is denoted by $d_{i,\ell,p}$. For example, $d_{\text{Monday},2,99} = 16$ and $d_{\text{Thursday},1,129} = 27$ according to Table 1. Ultimately, the manager seeks to maximize the revenue from guests staying in the hotel, given the demand forecast.

Length of Stay (nights)	1	2	3	1	2	3	1	2	3
Room Rate (per night)	\$79	\$79	\$79	\$99	\$99	\$99	\$129	\$129	\$129
Monday	45	36	9	20	16	4	15	12	3
Tuesday	72	24	24	42	14	14	24	8	8
Wednesday	77	22	21	56	16	8	42	12	6
Thursday	72	8	0	54	6	0	27	3	0
Friday	50	40	10	15	12	3	5	4	1

Table 1: Example of a demand forecast by day of week, length of stay, and rate.

Remarks: (i) Assume the hotel is empty on Monday; (ii) ignore any integrality considerations; (iii) ignore capacity considerations during the days following Friday (e.g., assume that 2-day reservations made on Friday can be honored on Saturday); (iv) consider the full revenue from bookings you make.

(a) Formulate a LP that finds an optimal booking policy for the hotel, i.e., the number of room requests to be accepted for each combination of fare and length of stay, in each day of the week.

Hint: Don't forget that some customers stay longer than others!

(b) The manager notices that rejecting some booking requests on Tuesday may have negative consequences for future revenues, because many of these travelers are important business customers. In fact, the manager estimated that each of the first 100 rejections (of any type of request) on Tuesday would result in roughly \$50 of "lost customer goodwill," and any subsequent rejection (i.e., beyond 100) would result in a \$75 goodwill loss. For example, rejecting 140 of the requests on Tuesday would generate a goodwill loss of $100 \cdot \$50 + 40 \cdot \$75$. Adjust the LP you derived in part (a) to account for the goodwill loss. Your final optimization model must still be an LP to receive full credit.

Solution.

(a) The decision variables will be $x_{i,\ell,p}$, which represent the number of room requests to be accepted for each combination of day, length of stay, and price. For a given i, ℓ, p combination, the revenue generated is $\ell p x_{i,\ell,p}$. Therefore the objective function that we would like to maximize is

$$\sum_{i,\ell,p} \ell p x_{i,\ell,p}.$$

The constraints are:

- Positivity: $x_{i,\ell,p} \geq 0$.
- Accepted requests should not exceed demand: $x_{i,l,p} \leq d_{i,l,p}$.
- Hotel capacity cannot be exceeded on any given day i: $\sum_p x_{i,1,p} + x_{i,2,p} + x_{i,3,p} + x_{i-1,2,p} + x_{i-1,3,p} + x_{i-2,3,p} \leq 200$. Note that we use $i-1$ and $i-2$ as a shorthand for one and two days previous to day i. Furthermore, if $i-1$ or $i-2$ are before Monday, the corresponding terms are dropped from the sum.

Combining the objective and constraints yields the following LP:

$$\begin{array}{ll} \text{maximize} & \sum_{i,\ell,p} \ell p x_{i,\ell,p} \\ \text{subject to} & x_{i,\ell,p} \geq 0 \\ & x_{i,\ell,p} \leq d_{i,\ell,p} \\ & \sum_p x_{i,1,p} + x_{i,2,p} + x_{i,3,p} + x_{i-1,2,p} + x_{i-1,3,p} + x_{i-2,3,p} \leq 200. \end{array}$$

(b) We will use $g_{\ell,p}$ to denote the goodwill loss for each ℓ, p pair. The explicit form of $g_{\ell,p}$ is shown below (r denotes the number of rejections):

$$g_{\ell,p}(r) = \begin{cases} 0, & r \leq 0 \\ -50r, & 0 \leq r \leq 100 \\ -75r + 2500, & r \geq 100 \end{cases}.$$

Since $g_{\ell,p}$ is a piecewise linear, decreasing function, it can be expressed as a pointwise minimization of linear functions:

$$g_{\ell,p}(r) = \min(0, -50r, -75r + 2500).$$

Now let $r_{\text{Tuesday},\ell,p} = d_{\text{Tuesday},\ell,p} - x_{\text{Tuesday},\ell,p}$ denote the number of rejections on Tuesday for each ℓ, p pair. The new objective is

$$\sum_{i,\ell,p} \ell p x_{i,\ell,p} + \sum_{\ell,p} \min(0, -50r_{\text{Tuesday},\ell,p}, -75r_{\text{Tuesday},\ell,p} + 2500).$$

This objective is not linear, but it can be made linear by introducing a variable $t_{\ell,p}$ for each goodwill loss term in the sum, and using epigraph form. This transformation leads to the objective

$$\sum_{i,\ell,p} \ell p x_{i,\ell,p} + \sum_{\ell,p} t_{\ell,p}$$

and new constraints

$$t_{\ell,p} \leq 0,\ t_{\ell,p} \leq -50r_{\text{Tuesday},\ell,p},\ t_{\ell,p} \leq -75r_{\text{Tuesday},\ell,p} + 2500.$$

In summary, the new LP is

$$
\begin{array}{ll}
\text{maximize} & \sum_{i,\ell,p} \ell p x_{i,\ell,p} + \sum_{\ell,p} t_{\ell,p} \\
\text{subject to} & x_{i,\ell,p} \geq 0 \\
& x_{i,\ell,p} \leq d_{i,\ell,p} \\
& \sum_p x_{i,1,p} + x_{i,2,p} + x_{i,3,p} + x_{i-1,2,p} + x_{i-1,3,p} + x_{i-2,3,p} \leq 200 \\
& r_{\text{Tuesday},\ell,p} = d_{\text{Tuesday},\ell,p} - x_{\text{Tuesday},\ell,p} \\
& t_{\ell,p} \leq 0 \\
& t_{\ell,p} \leq -50 r_{\text{Tuesday},\ell,p} \\
& t_{\ell,p} \leq -75 r_{\text{Tuesday},\ell,p} + 2500.
\end{array}
$$

Problem 2. Feasibility of portfolio optimization

We wish to solve a variant of the standard Markowitz portfolio optimization problem. Rather than putting the risk in the objective, we will constrain the risk using the quadratic constraint $x^T \Sigma x \leq \gamma$, where Σ is a fixed covariance matrix and γ is a risk parameter. The portfolio weights x must also satisfy a budget constraint given by $\sum_i x_i = 1$.

(a) When Σ is positive-definite, what is the smallest value of the risk parameter γ for which the problem is feasible?

(b) When Σ is singular, what is the smallest value of the risk parameter γ for which the problem is feasible?

Hint: Think about what happens when $\mathbf{1} \in \mathbf{range}(\Sigma)$ and what happens when $\mathbf{1} \notin \mathbf{range}(\Sigma)$.

Solution.

(a) The smallest value of γ is the optimal value $P^\star$ of the following optimization problem:

$$\begin{array}{lll} \text{(P)} & \text{minimize} & x^T \Sigma x \\ & \text{subject to} & \mathbf{1}^T x = 1. \end{array}$$

Since P is convex and has no inequality constraints, strong duality is satisfied.

Let ν be the dual variable corresponding to $\mathbf{1}^T x = 1$. The Lagrangian is

$$\mathcal{L}(x, \nu) = x^T \Sigma x + \nu(\mathbf{1}^T x - 1).$$

By minimizing over x, we obtain the dual function

$$g(\nu) = -\frac{1}{4}\mathbf{1}^T \Sigma^{-1} \mathbf{1} \nu^2 - \nu.$$

Since the dual function is unconstrained, it can be maximized by taking the gradient $\nabla g(\nu)$ and setting it to 0, which yields $\nu^\star = -2(\mathbf{1}^T \Sigma^{-1} \mathbf{1})^{-1}$. Therefore $P^\star = g(\nu^\star) = (\mathbf{1}^T \Sigma^{-1} \mathbf{1})^{-1}$.

(b) **Case 1: $\mathbf{1} \in$ range(Σ)**

Similar to part (a), we start by minimizing the Lagrangian $\mathcal{L}$ with respect to x. This yields the equation

$$2\Sigma x^\star = -\nu \mathbf{1}.$$

Although Σ is singular, the assumption $\mathbf{1} \in \textbf{range}(\Sigma)$ guarantees the existence of an $x^\star$ satisfying this equation—in fact, $x^\star = -(\nu \Sigma^+ \mathbf{1})/2$, where Σ^+ is the pseudoinverse of Σ.

Substituting $x^\star$ into the Lagrangian gives the dual function

$$g(\nu) = -\frac{1}{4}\mathbf{1}^T \Sigma^{+} \mathbf{1} \nu^2 - \nu.$$

Maximizing the dual function yields $g(\nu^\star) = (\mathbf{1}^T \Sigma^+ \mathbf{1})^{-1}$.

Case 2: $\mathbf{1} \notin$ range(Σ)

Since Σ is a covariance matrix, it is psd. Therefore $x^T \Sigma x \geq 0$ for all x, implying the minimal γ is ≥ 0.

Since $\mathbf{1} \notin \textbf{range}(\Sigma)$, we know that $\mathbf{1} \notin \textbf{null}(\Sigma)^\perp$. This means that there exists $x \in \textbf{null}(\Sigma)$ for which $\mathbf{1}^T x \neq 0$.

Let $x' = x/(\mathbf{1}^T x)$. Then $(x')^T \Sigma x' = x^T (\Sigma x)/(\mathbf{1}^T x)^2 = x^T 0/(\mathbf{1}^T x)^2 = 0$ and $\mathbf{1}^T x' = (\mathbf{1}^T x)/(\mathbf{1}^T x) = 1$. Thus the minimal γ is 0.

Problem 3. Dual norms

Let $\|\cdot\|$ denote any norm on $\mathbf{R}^n$. The *dual norm* $\|\cdot\|_*$ of this norm is defined as follows:

$$\|y\|_* = \begin{array}{ll} \text{maximize} & x^T y \\ \text{subject to} & \|x\| \leq 1. \end{array}$$

Note that in the above problem x is the vector of decision variables and the vector y is a given constant vector.

(a) Prove that for any two vectors $x, y \in \mathbf{R}^n$ we always have that $|x^T y| \leq \|x\|\|y\|_*$.

(b) Use LP duality to prove that the dual norm of the *max norm* $\|y\|_\infty \equiv \max_i\{|y_i|\}$ is the *1-norm* $\|y\|_1 \equiv \sum_i |y_i|$.

Solution.

(a) When $x = 0$ the result follows immediately. For the remainder of the proof, assume $x \neq 0$.

Let $\hat{x} = x/\|x\|$. Then $\|\hat{x}\| = 1$. Using the definition of the dual norm yields $\hat{x}^T y \leq \|y\|_*$. Multiplying by $\|x\|$ on both sides leads to $x^T y \leq \|x\|\|y\|_*$. By the same logic, $-x^T y \leq \|-x\|\|y\|_* = \|x\|\|y\|_*$. Therefore, $-\|x\|\|y\|_* \leq x^T y \leq \|x\|\|y\|_*$, which immediately implies $|x^T y| \leq \|x\|\|y\|_*$.

(b) By definition, the dual norm of $\|y\|_\infty$ is the optimal value of

$$\begin{array}{ll} \text{maximize} & x^T y \\ \text{subject to} & \|x\|_\infty \leq 1. \end{array}$$

This optimization problem can be written as an LP:

$$\begin{array}{ll} \text{maximize} & x^T y \\ \text{subject to} & -x - \mathbf{1} \leq 0 \\ & x - \mathbf{1} \leq 0. \end{array}$$

The dual of this LP is

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T(\lambda_1 + \lambda_2) \\ \text{subject to} & \lambda_2 - \lambda_1 = y \\ & \lambda_1 \geq 0 \\ & \lambda_2 \geq 0. \end{array}$$

The key insight is that this LP is *separable* over the components of the decision variables, so we can analyze a simpler problem for each component i:

$$\begin{array}{ll} \text{minimize} & \lambda_1^i + \lambda_2^i \\ \text{subject to} & \lambda_2^i - \lambda_1^i = y^i \\ & \lambda_1^i \geq 0 \\ & \lambda_2^i \geq 0. \end{array}$$

If we can show that the optimal value of this simpler problem is $|y^i|$, we are done.

Eliminating λ_2^i, we reach

$$\begin{array}{ll} \text{minimize} & 2\lambda_1^i + y^i \\ \text{subject to} & \lambda_1^i \geq 0 \\ & \lambda_1^i \geq -y^i. \end{array}$$

By inspection, we see $(\lambda_i^i)^\star = \max\{0, -y^i\}$. This implies that the optimal value of this LP is $2\max\{0, -y^i\} + y^i = \max\{y^i, -y^i\} = |y^i|$, which is exactly what we wanted to prove.