Duality - Part Three

October 9, 2024

Quick Announcements

- Homework 2 deadline extended until **Monday (Oct 14)**
- On the week of Oct 21 - Oct 25:
  - Class canceled on Monday (Oct 21)
  - Midterm exam on Wednesday (Oct 23)
  - Regular lecture on Friday (Oct 25)
- Some of you asked about homework : weight is 40% (as posted)
- I am posting slightly incomplete slides *on purpose* (complete after class)
- My office hours **today: 5-6pm**
- My office hours **starting next week: Wed, 3-4pm**

Recap From Last Time & Today's Plan

Last time...

- **Separating Hyperplane Thm $\Rightarrow$ Farkas Lemma $\Rightarrow$ Strong duality**
- Implications on primal/dual feasibility
- Two examples (robust optimization, CVaR)

Agenda for today:

- Optimality conditions and primal/dual simplex
- Complementary slackness
- Representation Theorem for all polyhedra
- Global sensitivity & Shadow prices as marginal costs
- Two new applications (asset pricing and network revenue management)

Optimality for Standard-Form LPs

$$(\mathscr{P}) \ \min\ c^\intercal x \qquad\qquad (\mathscr{D}) \ \max\ p^\intercal b$$
$$Ax = b, \quad x \geq 0 \qquad\qquad p^\intercal A \leq c^\intercal$$

- $(\mathscr{P})$ achieves optimality at a **basic feasible solution** x:
  - If $B \subseteq \{1, \ldots, n\}$ is a basis, the b.f.s. is: $x = [x_B, 0], \quad x_B = A_B^{-1} b$.
  - Simplex algorithm: feasibility and optimality for $(\mathscr{P})$ are given by:

$$\text{Feasibility-}(\mathscr{P}): \qquad x_B := A_B^{-1} b \geq 0 \tag{1a}$$

$$\text{Optimality-}(\mathscr{P}): \qquad c^\intercal - c_B^\intercal A_B^{-1} A \geq 0 \tag{1b}$$

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- $(\mathscr{D})$: same basis B can also be used to determine **a dual vector** p:
$$p^\mathsf{T} A_i = c_i, \forall\, i \in B \quad \Rightarrow \quad p^\mathsf{T} = c_B^\mathsf{T} A_B^{-1}, \forall\, i \in B.$$
  - The dual objective value of p is exactly: $p^\mathsf{T} b = c_B^\mathsf{T} A_B^{-1} b = c^\mathsf{T} x$
  - p is feasible in the dual if and only if:

$$\text{Feasibility-}(\mathscr{D}): \qquad c^\mathsf{T} - p^\mathsf{T} A \geq 0 \quad \Leftrightarrow \quad c^\mathsf{T} - c_B^\mathsf{T} A_B^{-1} A \geq 0 \tag{2}$$

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Primal optimality $\Leftrightarrow$ Dual feasibility

Simplex terminates when finding a dual-feasible solution!

Solve $(\mathscr{P})$ or $(\mathscr{D})$?

$$(\mathscr{P}) \quad \min\ c^\intercal x \qquad Ax = b, \quad x \geq 0$$

$$(\mathscr{D}) \quad \max\ p^\intercal b \qquad p^\intercal A \leq c^\intercal$$

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Primal simplex

- maintain a **basic feasible solution**
- basis $B \subset \{1, \ldots, n\}$
- stopping criterion: dual feasibility

Dual simplex

- maintain a dual feasible solution
- stopping criterion: primal feasibility
- different from primal simplex: works with an LP with inequalities

- How to choose $(\mathscr{P})$ or $(\mathscr{D})$?
- Suppose we have x^*, p^* and must solve a **larger** problem. *Any ideas?*

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- How to choose $(\mathscr{P})$ or $(\mathscr{D})$?
- Suppose we have x^*, p^* and must solve a **larger** problem. *Any ideas?*
  - With extra decisions $x_e \Rightarrow$ **primal simplex** initialized with $[x^*, x_e = 0]$.
  - With extra constraints $A_e x = b_e \Rightarrow$ **dual simplex** initialized with $[p^*, p_e = 0]$.
- Modern solvers include primal and dual simplex and allow concurrent runs

Optimality Conditions and Complementary Slackness

$$
\begin{array}{llll}
\min_x & c^\top x & & \\
& a_i^\top x \geq b_i, & i \in M_1, \\
& a_i^\top x \leq b_i, & i \in M_2, \\
& a_i^\top x = b_i, & i \in M_3, \\
& x_j \geq 0, & j \in N_1, \\
& x_j \leq 0, & j \in N_2, \\
& x_j \text{ free}, & j \in N_3.
\end{array}
\qquad
\begin{array}{llll}
\max_p & p^\top b & & \\
& p_i \geq 0, & i \in M_1, \\
& p_i \leq 0, & i \in M_2, \\
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& p^\top A_j \geq c_j, & j \in N_2, \\
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Sometimes, we just want to **characterize** the optimal solutions

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Sometimes, we just want to **characterize** the optimal solutions

Theorem (Complementary Slackness)

Let x and p be feasible solutions for $(\mathscr{P})$ and $(\mathscr{D})$, respectively. Then x and p **are optimal solutions** *for $(\mathscr{P})$ and $(\mathscr{D})$* **if and only if***:*

$$p_i(a_i^\intercal x - b_i) = 0, \ \forall i$$
$$(c_j - p^\intercal A_j)x_j = 0, \ \forall j.$$

Optimality Conditions and Complementary Slackness

Theorem (**General** Complementary Slackness)

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Theorem (**Strict** C.S. **Standard-Form LPs**)

Consider the following primal-dual pair of LPs:

$$(\mathscr{P}) \ \min \ c^\intercal x \qquad\qquad (\mathscr{D}) \ \max \ p^\intercal b$$
$$Ax = b, x \geq 0 \qquad\qquad p^\intercal A \leq c^\intercal$$

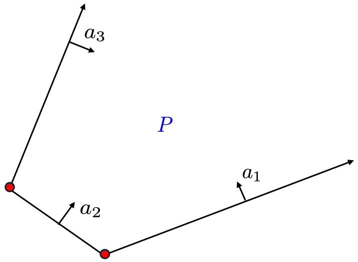
If $(\mathscr{P})$ and $(\mathscr{D})$ are feasible, they admit optimal solutions x^* and p^* satisfying **strict complementarity***: $\ x_j^* > 0 \ \Leftrightarrow \ p^\intercal A_j = c_j$.*

Representation of Polyhedra

Important consequence of duality: alternative representation of all polyhedra

Definition

Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$. Then:



Representation of Polyhedra

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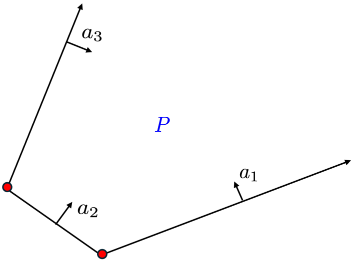
Definition

Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$. Then:

1. $\mathcal{C} := \{d \in \mathbb{R}^n : Ad \geq 0\}$ is called the **recession cone** of P.
2. Any $d \in \mathcal{C}$ with $d \neq 0$ is called a **ray** of P.



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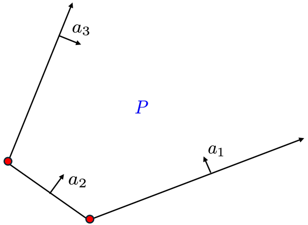
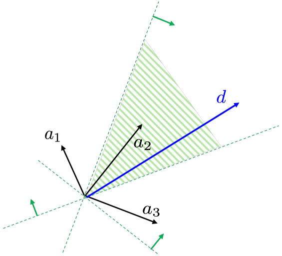
2. Any $d \in \mathcal{C}$ with $d \neq 0$ is called a **ray** of P.
3. Any ray d that satisfies $a_i^\intercal d = 0$ for $n-1$ linearly independent a_i is called an **extreme ray** of P.





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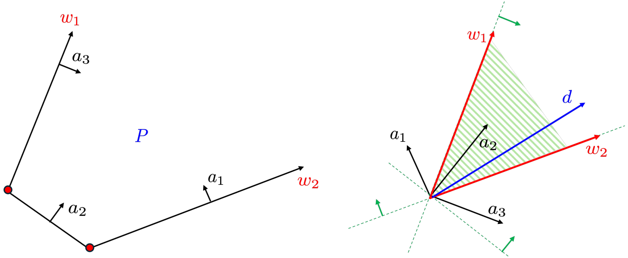

Representation of Polyhedra

Theorem (Resolution Theorem)

Let $P = \{x \in \mathbb{R}^n \;:\; Ax \geq b\}$ *be a non-empty polyhedron,* $x^1, x^2, \ldots, x^k$ *be its extreme points, and* $w^1, w^2, \ldots, w^r$ *be its extreme rays. Then* $P = Q$*, where*

$$Q := \left\{ \sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j \;:\; \lambda \geq 0,\ \theta \geq 0,\ e^\mathsf{T}\lambda = 1 \right\}.$$

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Proof. Proving $Q \subseteq P$ is immediate. To prove $P \subseteq Q$, assume $\exists z \in P$ with $z \notin Q$. Consider the following primal-dual pair:

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$$(\mathscr{P}) \quad \max_{\lambda \geq 0, \theta \geq 0} \ \sum_{i=1}^{k} 0\lambda_i + \sum_{j=1}^{r} 0\theta_j$$

$$\sum_{i=1}^{k} \lambda_i x^i + \sum_{j=1}^{r} \theta_j w^j = z$$

$$\sum_{i=1}^{k} \lambda_i = 1$$

$$(\mathscr{D}) \quad \min_{p,q} \ p^\top z + q$$

$$p^\top x_i + q \geq 0, \quad i = 1, \ldots, k,$$

$$p^\top w_j \geq 0, \quad j = 1, \ldots, r,$$

Is $(\mathscr{P})$ feasible? Is $(\mathscr{D})$ feasible? What are the optimal values?

Representation of Polyhedra - cntd

$P := \{x \in \mathbb{R}^n : Ax \geq b\} = Q := \left\{ \sum_{i=1}^k \lambda_i x^i + \sum_{j=1}^r \theta_j w^j : \lambda \geq 0, \theta \geq 0, e^\intercal \lambda = 1 \right\}.$

Proof - cont'd. Assume $\exists z \in P$ with $z \notin Q$. Consider the following primal-dual pair:

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$$(\mathscr{D}) \quad \min_{p,q} p^\intercal z + q$$
$$p^\intercal x_i + q \geq 0, \quad i = 1, \ldots, k,$$
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- $(\mathscr{P})$ is infeasible because $z \notin Q$
- $(\mathscr{D})$ is feasible with $p = q = 0$, so its optimal value is $-\infty \Rightarrow \exists (p, q) : p^\intercal z + q < 0$

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- (p,q) feasible $\Rightarrow p^\intercal z < -q \leq p^\intercal x_i$ for any $i = 1, \ldots, k$ and $p^\intercal w_i \geq 0$
- With p as above, consider the LP $\min_x \{p^\intercal x : Ax \geq b\}$

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- With p as above, consider the LP $\min_x \{p^\mathsf{T} x : Ax \geq b\}$
- If optimal cost finite, $\exists x^i$ optimal. But $z \in P$ and $p^\mathsf{T} z < p^\mathsf{T} x_i$ lead to $\lightning$
- If cost is $-\infty$, $\exists w^j : p^\mathsf{T} w^j < 0$, which is also a $\lightning$

Dual Variables **As Marginal Costs**

$$(\mathscr{P}) \;\; \min\, c^\intercal x \qquad\qquad (\mathscr{D}) \;\; \max\, p^\intercal b$$

$$Ax = b, \quad x \geq 0 \qquad\qquad p^\intercal A \leq c^\intercal$$

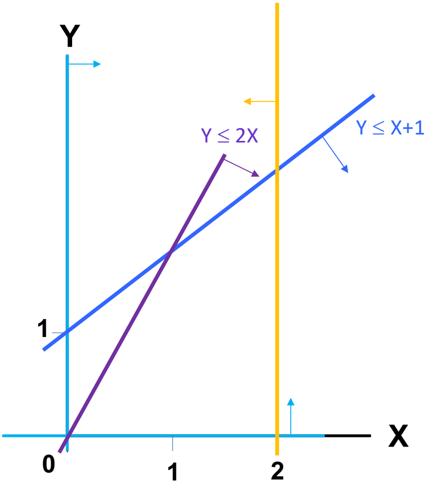
- Solved the LP and obtained x^* and p^*
- Want to show that p^* is **gradient of the optimal cost with respect to** b ("almost everywhere")
- Related to **sensitivity analysis**
  How do the optimal value and solution depend on problem data A, b, c*?*

Sensitivity: A Simple Example

Maximize **Y**

Subject to:

$Y \leq 2X$
$Y \leq X+1$
$X \geq 0, Y \geq 0$
$X \leq 2$



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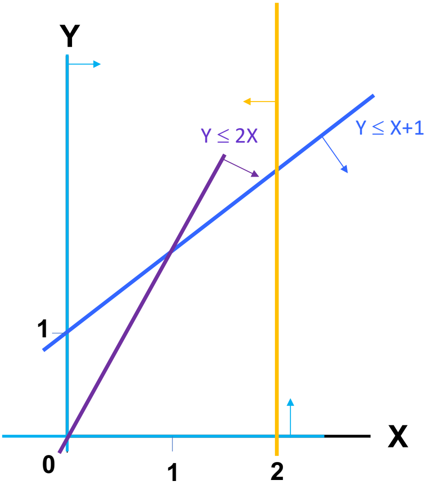
$\mathbf{X \leq a}$

For the last constraint $X \leq a$, what is the shadow price i.e., rate of change in the optimal value when we change the constraint r.h.s. a?



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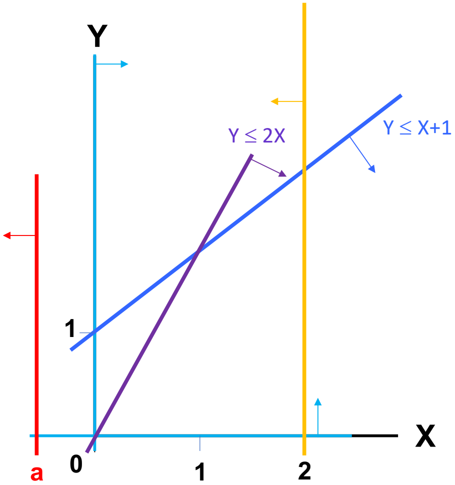
If $a < 0$:



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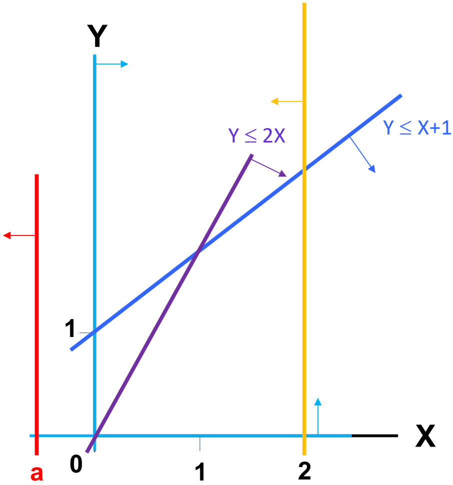
If a < 0 :
- **Infeasible!**



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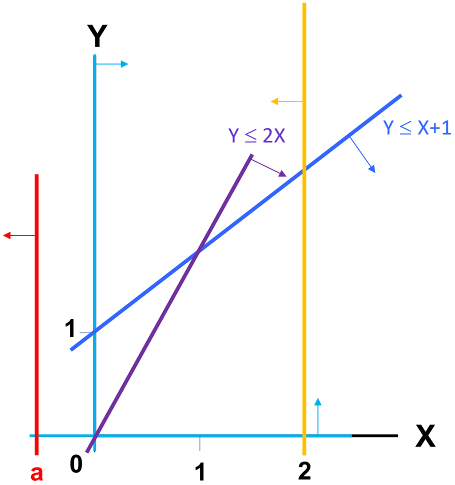

If a < 0 :

- **Infeasible!**
- Shadow price = $+\infty$

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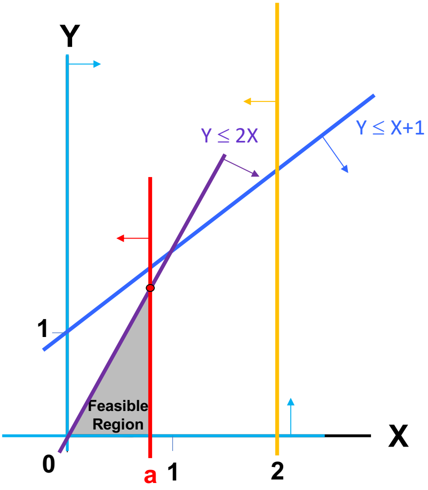
If $0 < a < 1$:



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Subject to:
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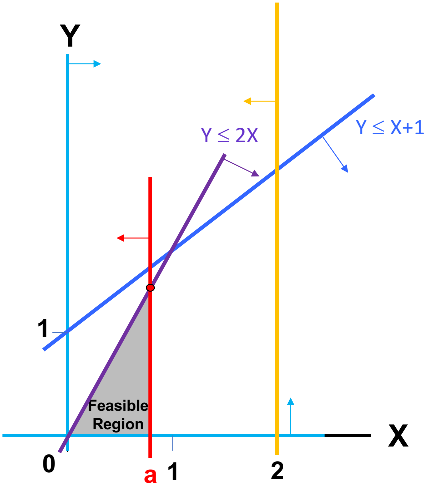
If $0 < a < 1$:

- Shadow price = 2



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If 0 < a < 1 :

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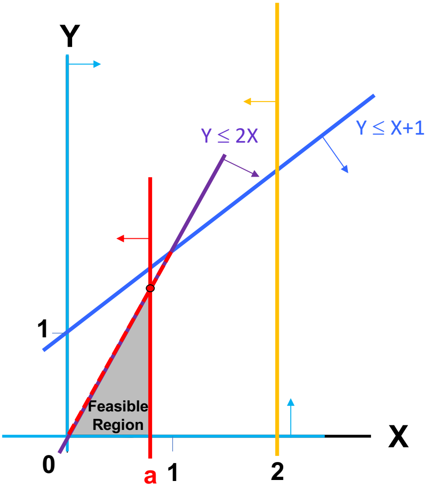

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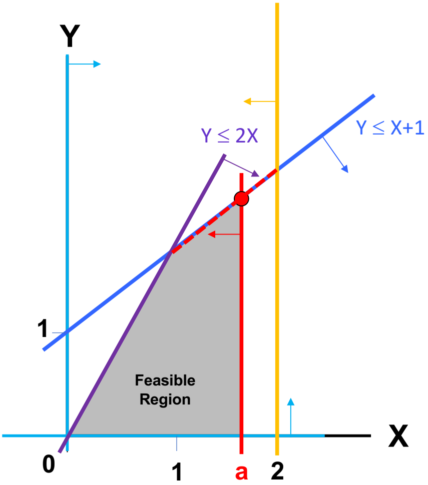
If $1 < a < 2$:



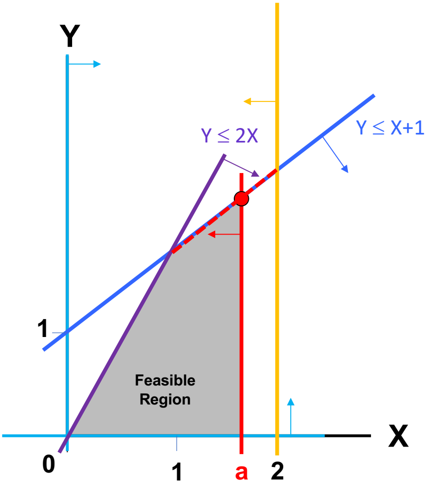
Sensitivity: A Simple Example

Maximize **Y**

Subject to:
$Y \leq 2X$
$Y \leq X+1$
$X \geq 0, Y \geq 0$
$X \leq 2$
$X \leq a$

If 1 < a < 2 :

- **Shadow price = 1**



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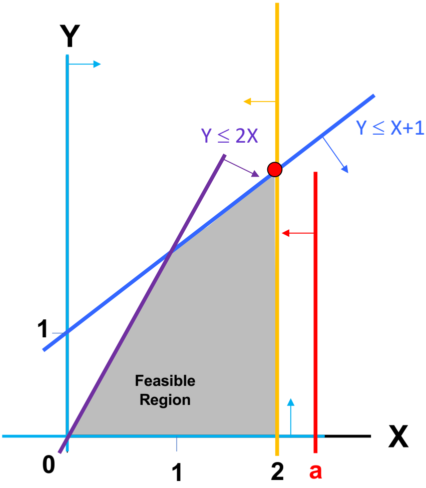
If $a > 2$:



Global Dependency On b

$$(\mathscr{P}) \ \min c^\mathsf{T} x \qquad Ax = b, \quad x \geq 0$$

$$(\mathscr{D}) \ \max p^\mathsf{T} b \qquad p^\mathsf{T} A \leq c^\mathsf{T}$$

- Let $P(b) := \{x : Ax = b, x \geq 0\}$ and $F(b)$ denote the optimal cost
- Assume that dual is feasible: $\{p : p^\mathsf{T} A \leq c^\mathsf{T}\} \neq \emptyset$, so $F(b) > -\infty$
- Want to show that $F(b)$ is **piecewise linear and convex**

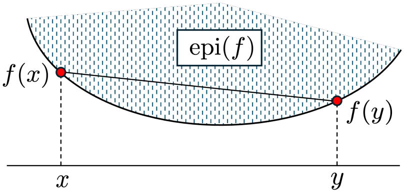
Convex and Concave Functions

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **convex** if X is a convex set and

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in X \text{ and } \lambda \in [0,1]. \tag{3}$$

A function is **concave** if $-f$ is convex.



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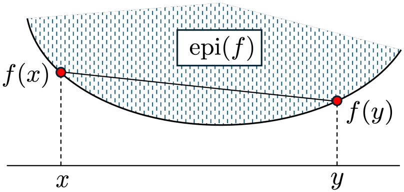

Equivalent definition in terms of **epigraph**:

$$\text{epi}(f) = \{(x,t) \in X \times \mathbb{R} \,:\, t \ge f(x)\} \tag{4}$$

f is convex if and only if epi(f) is a convex set.

Global Dependency On b

$$F(b) := \min\left\{c^\intercal x : Ax = b,\ x \geq 0\right\} \equiv \max\left\{p^\intercal b \,:\, p^\intercal A \leq c^\intercal\right\}$$

Theorem

$F(b)$ *is a convex and piece-wise linear function of* b *on* $S := \{b : P(b) \neq \emptyset\}$.

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Proof. Claim: S is convex. Why?

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Proof. Claim: S is convex. Why?

It is the cone spanned by the columns of A

$$S := \text{cone}(\{A_1, A_2, \ldots, A_n\})$$

Recall that we dealt with this same cone in the proof of the Farkas Lemma!

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Theorem

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Proof. Because $(\mathscr{D})$ feasible $\Rightarrow$ $F(b) > -\infty$.

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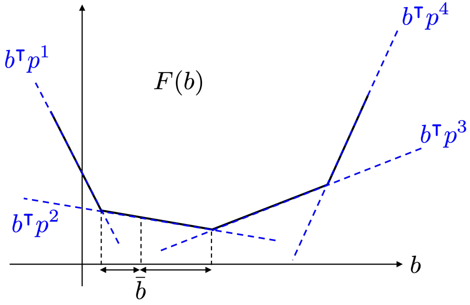
- If $p^1, p^2, \ldots, p^r$ are the extreme points of the dual feasible set, then: $F(b) = \max_{i=1,\ldots,r} b^\intercal p^i$, $\forall\, b \in S$



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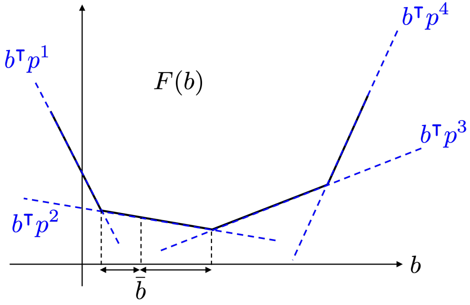

How to complete proof that $F(b)$ is convex?

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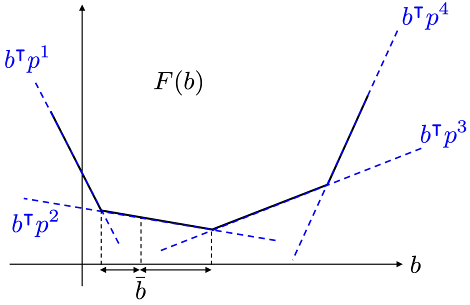

How to complete proof that $F(b)$ is convex?

$$\text{epi}(F) = \cap_{i=1,\ldots,r}\text{epi}(b^\intercal p^i)$$

is the intersection of convex sets, so it is convex.

Global Dependency On b - Implications

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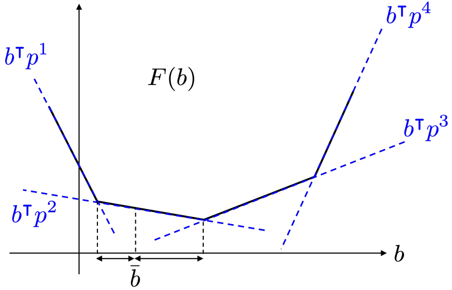

- At any b where $F(b)$ is differentiable, p^* **is the gradient of $F(b)$**
- p_i^* acts as a **marginal cost** or **shadow price** for the i-th constraint r.h.s. b_i
- p_i allows estimating **exact change in $F(b)$ in a range around $\bar{b}$**

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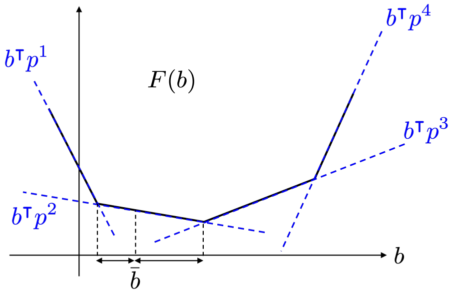

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- p_i^* acts as a **marginal cost** or **shadow price** for the i-th constraint r.h.s. b_i
- p_i allows estimating **exact change in $F(b)$ in a range around $\bar{b}$**
- Modern solvers give **direct access to p_i^* and the range**

  Gurobipy: for constraint **c**, the attribute **c.Pi** is p_i^* and the range is from **c.SARHSLow** to **c.SARHSUp**

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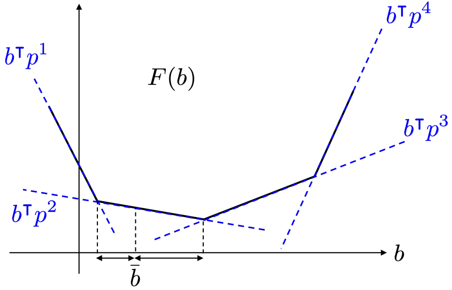

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- All such p^i are valid **subgradients** of $F(b)$

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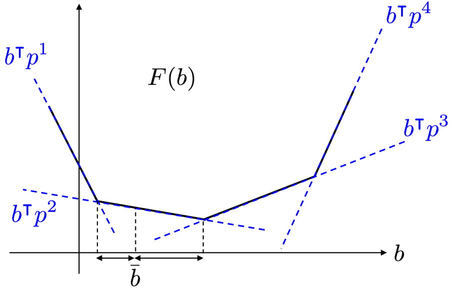

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Definition (Subgradient.)

F convex, defined on (convex) set S. A vector p is a **subgradient** of F at $\bar{b} \in S$ if

$$F(\bar{b}) + p^\intercal(b - \bar{b}) \leq F(b), \quad \forall b \in S.$$

Optimal Duals As Subgradients

Theorem

Suppose $F(b) := \min\{c^\intercal x : Ax = b,\ x \geq 0\} \equiv \max\{p^\intercal b \,:\, p^\intercal A \leq c^\intercal\} > -\infty$. *Then* p *is optimal for the dual* **if and only if** *it is a subgradient of* F *at* $\bar{b}$.

Proof. First show that any dual optimal p is a valid subgradient.

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- For any feasible solution $x \in P(b)$, weak duality yields $p^\intercal b \leq c^\intercal x$
- This implies $p^\intercal b \leq F(b)$

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- This implies $p^\intercal b \leq F(b)$
- But then, $p^\intercal b - p^\intercal \bar{b} \leq F(b) - F(\bar{b})$

We conclude that p is a subgradient of F at $\bar{b}$

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Proof. For the reverse direction, let p be a subgradient of F at $\bar{b}$, that is,

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- This implies that p is dual-feasible
- With $x = 0$, we obtain $F(\bar{b}) \leq p^\intercal \bar{b}$
- Using weak duality, every dual-feasible q satisfies $q^\intercal \bar{b} \leq F(\bar{b}) \leq p^\intercal \bar{b}$

We conclude that p is optimal.

Global Dependency On c

Let $G(c) := \min\{c^\intercal x : Ax = b,\ x \geq 0\} \equiv \max\{p^\intercal b\ :\ p^\intercal A \leq c^\intercal\}$

Theorem

For an LP in standard form,

1. *The set* $T := \{c : G(c) > -\infty\}$ *is convex.*
2. $G(c)$ *is a* **concave** *function of* c *on the set* T*.*
3. *If for some* c *the LP has a* **unique** *optimal solution* x^**, then* G *is linear in the vicinity of* c *and its gradient is* x^**.*

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Proof. Analogous ideas applied to the dual - omitted.

- The optimal primal solution x^* **is a shadow price for the dual constraints**
- x^* remains optimal for a range of change in each objective coefficient c_j
- Modern solvers also allow obtaining the range directly

  Gurobipy: attributes **SAObjLow** and **SAObjUp** for each decision variable

Signs of Dual Variables Revisited

These ideas carry over directly to **primal-dual pairs in general form**:

$$
\begin{array}{llllll}
F(b,c) := \min_x & c^\intercal x & & \max_p & p^\intercal b & \\
& a_i^\intercal x \geq b_i, & i \in M_1, & & p_i \geq 0, & i \in M_1, \\
& a_i^\intercal x \leq b_i, & i \in M_2, & & p_i \leq 0, & i \in M_2, \\
& a_i^\intercal x = b_i, & i \in M_3, & & p_i \text{ free}, & i \in M_3, \\
& x_j \geq 0, & j \in N_1, & & p^\intercal A_j \leq c_j, & j \in N_1, \\
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- $F(b,c)$ is piece-wise linear, convex in b and piece-wise linear, concave in c
- p^* is subgradient for $F(b,c)$ with respect to b
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  - the **constraint type** ($\leq$, $\geq$)
  - **the signs of the shadow prices**

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  - **the optimization problem** (max/min)
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- Given two of these, can figure out the third one!
- *What is the sign of the shadow price for a ...*

  $\leq$ constraint in a **minimization** problem ?

  $\geq$ constraint in a **minimization** problem ?

  $\leq$ constraint in a **maximization** problem ?

  $\leq$ constraint in a **maximization** problem ?
- *What is the dependency of the optimal objective on the r.h.s. of a ...*

  $\leq$ constraint in a **minimization** problem ?

  $\geq$ constraint in a **minimization** problem ?

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- There is a direct connection between:
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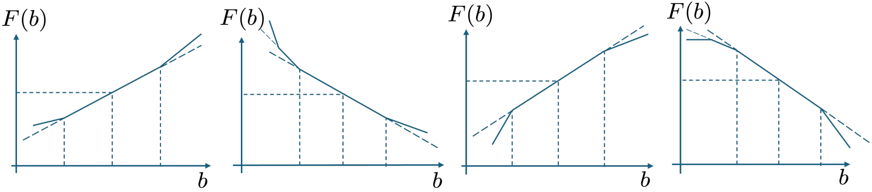
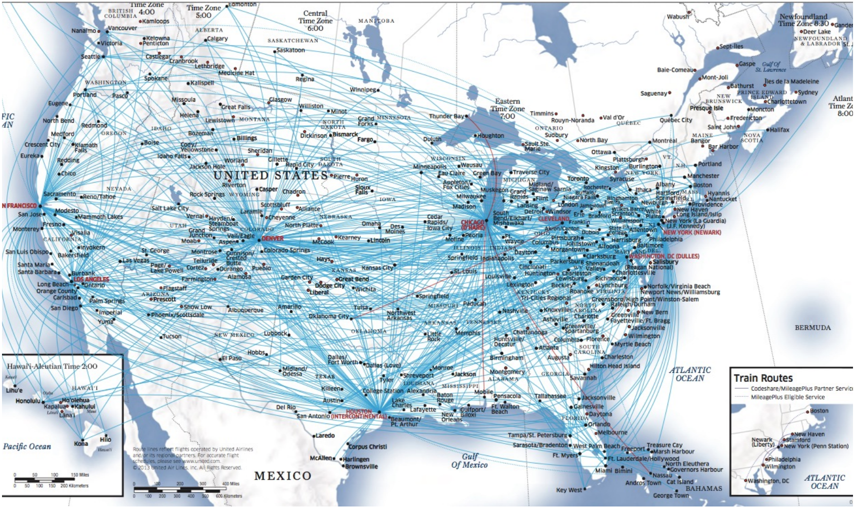
$\min, \geq b$	$\min, \leq b$	$\max, \leq b$	$\max, \geq b$
dual ≥ 0	dual ≤ 0	dual ≥ 0	dual ≤ 0
$F(b)$ convex	$F(b)$ convex	$F(b)$ concave	$F(b)$ concave



Real-World Hub and Spoke Airline Network



Source: www.united.com

Airline Revenue Management (RM)

- **Strategic RM**
  - Determine several price points for various itineraries
  - “Product” or “itinerary”: origin, destination, day, time, various restrictions, …
    - E.g., JFK – ORD – SFO, 10:30am on Oct 12, 2024, Economy class Y fare
  - Typically done by (or in conjunction with) marketing department
    - Market segmentation; competition

- **Tactical RM** (“yield management”) decides **booking limits**
  - A *booking limit* determines how many seats to reserve for each “product”
  - RM not based on setting prices, but rather changing availability of fare classes
  - Legacy due to original IT systems used (e.g., SABRE)

Airline RM

Hub: Chicago ORD

Two planes

Westbound flights for some day in the future

BOS

SFO

ORD

LAX

JFK

Airline RM

Flight segments (legs)

SFO

BOS

ORD

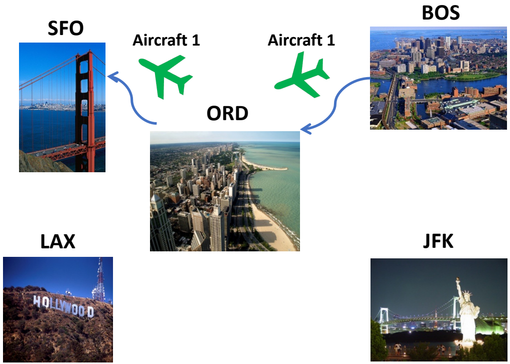
LAX

JFK

Airline RM

Flight segments (legs)

- **Aircraft 1:**
  - BOS-ORD in the morning
  - ORD-SFO in the afternoon



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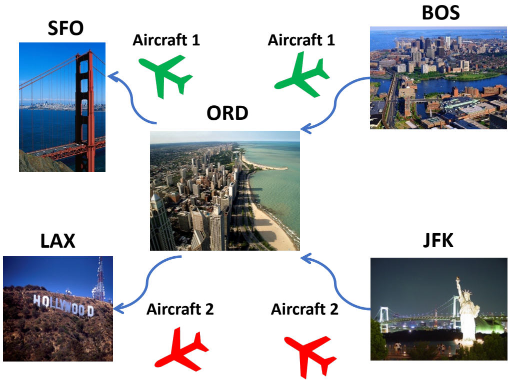

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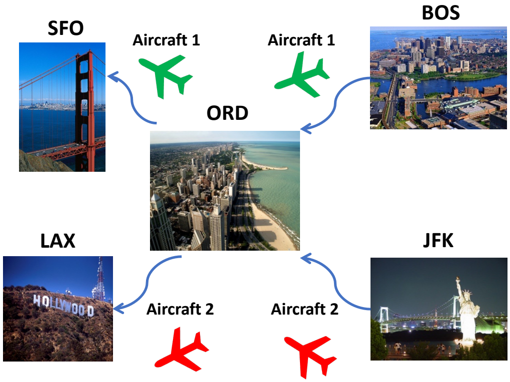

Itineraries

Origin-Destination	Q_Fare	Y_Fare
BOS_ORD	$200	$220
BOS_SFO	$320	$420
BOS_LAX	$400	$490
JFK_ORD	$250	$290
JFK_SFO	$410	$540
JFK_LAX	$450	$550
ORD_SFO	$210	$230
ORD_LAX	$260	$300

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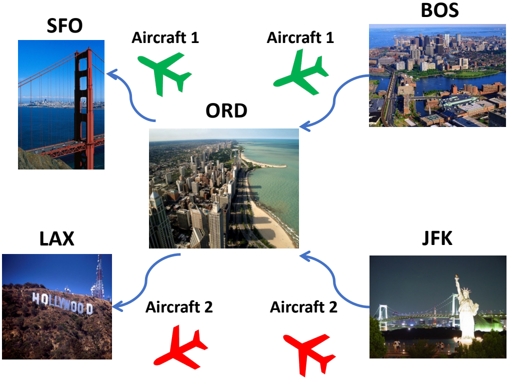

Itineraries

Origin-Destination	Q_Fare	Y_Fare	Q_Demand	Y_Demand
BOS_ORD	$200	$220	25	20
BOS_SFO	$320	$420	55	40
BOS_LAX	$400	$490	65	25
JFK_ORD	$250	$290	24	16
JFK_SFO	$410	$540	65	50
JFK_LAX	$450	$550	40	35
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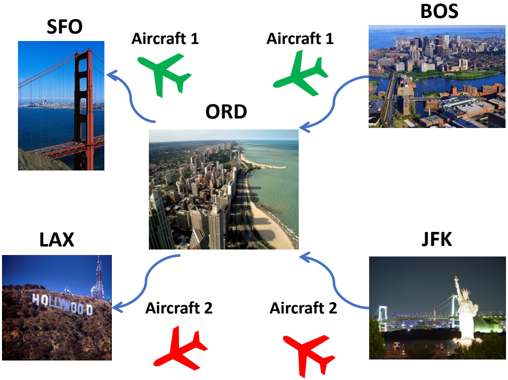

Resources needed

Flight leg	BOS_ORD	BOS_SFO	BOS_LAX	JFK_ORD	JFK_SFO	JFK_LAX	ORD_SFO	ORD_LAX
BOS_ORD_Leg	1	1	1	0	0	0	0	0
JFK_ORD_Leg	0	0	0	1	1	1	0	0
ORD_SFO_Leg	0	1	0	0	1	0	1	0
ORD_LAX_Leg	0	0	1	0	0	1	0	1

Network Revenue Management

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  - each itinerary refers to an origin-destination-fare class combination
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- Requirements: $A \in \{0,1\}^{F \cdot I}$ with $A_{f,i} = 1 \Leftrightarrow$ itinerary i needs seat on flight leg f

Resource matrix A :

	Itinerary 1	Itinerary 2	...	Itinerary $\lvert I\rvert$
Flight leg 1	1	0	...	1
Flight leg 2	0	1	...	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Flight leg $\lvert F\rvert$	1	1	...	0

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Flight leg $\|F\|$	1	1	$\ldots$	0

- Goal: decide how many itineraries of each type to sell to maximize revenue

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- **Bid-price heuristic** in network revenue management
- Broader principle of how to price "products" through resource usage/cost

Asset Pricing and No-Arbitrage

- Investment world with $n+1$ securities indexed by $i = 0, \ldots, n$
- $i = 0$ denotes cash; the other securities can be anything (stocks, derivatives, ...)
- We have two periods: current period **c**, future period **f**

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  - cash is riskless: $S_0^f = R = 1 + r$, where r is the risk-free rate of return
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- If we purchase x_i of each security i:
  - we incur immediate cost $\sum_{i=0}^n S_i^c x_i$
  - we have future cashflow $\sum_{i=0}^n S_i^f(\omega) \cdot x_i$ if state of world is $\omega \in \Omega$

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Definition (Arbitrage)

An **arbitrage** is a trading strategy that either has a positive initial cashflow and has no risk of a loss later (type A) or that requires no initial cash input, has no risk of loss, and has a positive probability of making profits in the future (type B).

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$$\begin{aligned} \sum_{i=0}^{n} S_i^c \cdot x_i < 0 & \qquad \text{(positive initial cashflow)} \\ \sum_{i=0}^{n} S_i^f(\omega) \cdot x_i \geq 0, \forall \omega \in \Omega & \qquad \text{(no risk of loss)} \end{aligned} \tag{6}$$

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Asset Pricing and No-Arbitrage

Definition (R.N.P.M.)

A **risk-neutral probability measure** on the set $\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ is a vector $p \in \mathbb{R}^m$ so that $p > 0$ and $\sum_{j=1}^m p_j = 1$ and for every security $S_i, i = 0, \ldots, n$,

$$S_i^c = \frac{1}{R}\left(\sum_{j=1}^m p_j S_i^f(\omega_j)\right) = \frac{1}{R}\mathbb{E}_p[S_i^f].$$

- Above, $\mathbb{E}_p[S]$ is the expected value of the random variable S under the probability distribution $p := (p_1, p_2, \ldots, p_m)$
- The definition states that the current price/value of every asset, S_i^c, exactly equals **the discounted expected price/value in the future**
- The expectation is taken with respect to the R.N.P.M.
- Discounting is done at the risk-free interest rate R

Asset Pricing and No-Arbitrage

Theorem (Asset Pricing Theorem)

A risk-neutral probability measure exists **if and only if** *there is no arbitrage.*

Proof. Consider the following linear program with variables x_i, for $i = 0, \ldots, n$:

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- No type-A arbitrage if and only if the optimal objective value of this LP is 0

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- Suppose no type-A arbitrage. Then, no type-B arbitrage if and only if all constraints are tight for all optimal solutions of (8): $\sum_{i=0}^{n} S_i^f(\omega_j) \cdot x_i^* = 0$, for $j = 1, \dots, m$

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Proof. Consider the dual of this LP.

$$\max_{p} 0$$
$$\text{s.t. } \sum_{j=1}^{m} p_j \cdot S_i^f(\omega_j) = S_i^c,\ i = 0, \ldots, n,$$
$$p_j \geq 0.$$

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- Dual constraint for $i = 0$ implies $\sum_{j=1}^{m} p_j^* = \frac{1}{R}$, so taking $p^* \cdot R$ yields a RNPM.

The converse direction is proved in an identical manner. ■