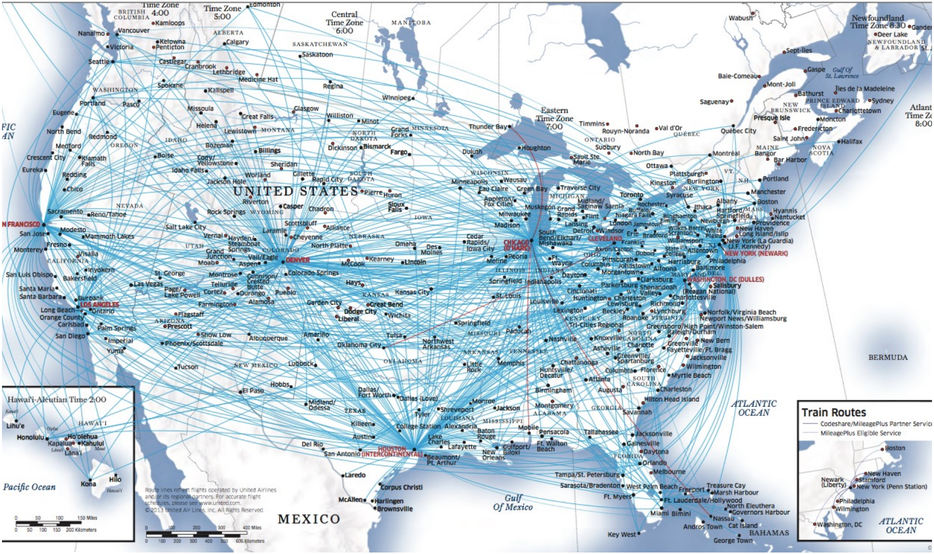
Lecture 7

October 14, 2024

Real-World Hub and Spoke Airline Network



Source: www.united.com

Airline Revenue Management (RM)

- **Strategic RM**
  - Determine several price points for various itineraries
  - “Product” or “itinerary”: origin, destination, day, time, various restrictions, …
    - E.g., JFK – ORD – SFO, 10:30am on Oct 12, 2024, Economy class Y fare
  - Typically done by (or in conjunction with) marketing department
    - Market segmentation; competition

- **Tactical RM** (“yield management”) decides **booking limits**
  - A *booking limit* determines how many seats to reserve for each “product”
  - RM not based on setting prices, but rather changing availability of fare classes
  - Legacy due to original IT systems used (e.g., SABRE)

Airline RM

Hub: Chicago ORD

Two planes

Westbound flights for some day in the future

SFO

BOS

ORD

LAX

JFK

Airline RM

Flight segments (legs)

SFO

BOS

ORD

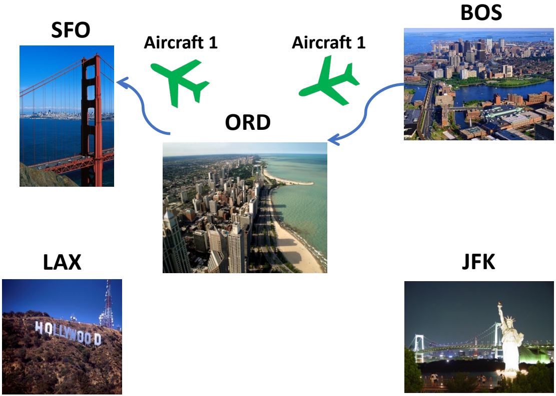
LAX

JFK

Airline RM

Flight segments (legs)

- **Aircraft 1:**
  - BOS-ORD in the morning
  - ORD-SFO in the afternoon



Airline RM

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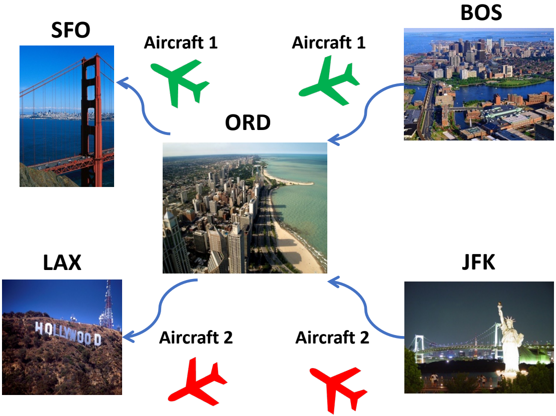

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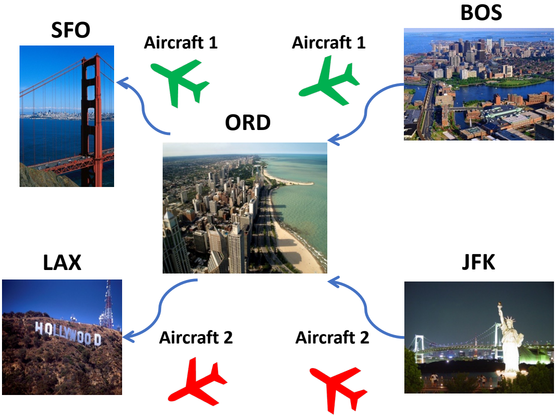

Itineraries

Origin-Destination	Q_Fare	Y_Fare
BOS_ORD	$200	$220
BOS_SFO	$320	$420
BOS_LAX	$400	$490
JFK_ORD	$250	$290
JFK_SFO	$410	$540
JFK_LAX	$450	$550
ORD_SFO	$210	$230
ORD_LAX	$260	$300

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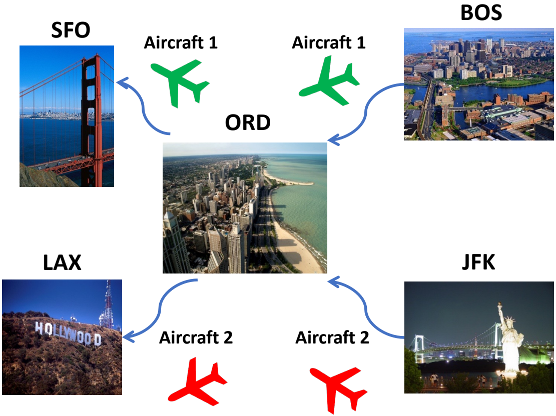

Itineraries

Origin-Destination	Q_Fare	Y_Fare	Q_Demand	Y_Demand
BOS_ORD	$200	$220	25	20
BOS_SFO	$320	$420	55	40
BOS_LAX	$400	$490	65	25
JFK_ORD	$250	$290	24	16
JFK_SFO	$410	$540	65	50
JFK_LAX	$450	$550	40	35
ORD_SFO	$210	$230	21	50
ORD_LAX	$260	$300	25	14

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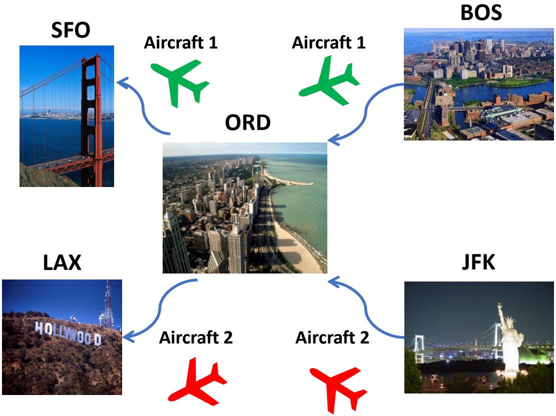

Resources needed

	BOS_ORD	BOS_SFO	BOS_LAX	JFK_ORD	JFK_SFO	JFK_LAX	ORD_SFO	ORD_LAX
Flight leg								
BOS_ORD_Leg	1	1	1	0	0	0	0	0
JFK_ORD_Leg	0	0	0	1	1	1	0	0
ORD_SFO_Leg	0	1	0	0	1	0	1	0
ORD_LAX_Leg	0	0	1	0	0	1	0	1

Network Revenue Management

- Airline is planning operations for a specific day in the future
- Airline operates a set F of direct flights in its (hub-and-spoke) network
- For each flight leg $f \in F$, we know the capacity of the aircraft c_f
- The airline can offer a large number of "products" (i.e., itineraries) I:
  - each itinerary refers to an origin-destination-fare class combination
  - each itinerary i has a price r_i that is fixed
  - for each itinerary, the airline estimates the demand d_i
  - each itinerary requires a seat on several flight legs operated by the airline

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- Requirements: $A \in \{0,1\}^{F \cdot I}$ with $A_{f,i} = 1 \Leftrightarrow$ itinerary i needs seat on flight leg f

Resource matrix A :

	Itinerary 1	Itinerary 2	...	Itinerary $\|I\|$
Flight leg 1	1	0	...	1
Flight leg 2	0	1	...	0
⋮	⋮	⋮	⋮	⋮
Flight leg $\|F\|$	1	1	...	0

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- Goal: decide how many itineraries of each type to sell to maximize revenue

Network Revenue Management

- x_i: number of itineraries of type i that the airline plans to sell
- Airline Network RM problem:

$$\max_{x \in \mathbb{R}^I} \left\{ r^\intercal x : Ax \leq c,\ x \leq d \right\}$$

- $Ax \leq c$: constraints on plane capacity
- $x \leq d$: planned sales cannot exceed the demand
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- To sell "exotic itineraries", use the **shadow prices for the capacity constraints**
  - $p \in \mathbb{R}^F$: dual variables for capacity constraints $Ax \leq c$
  - At optimality, p_f is marginal revenue lost if airline loses one seat on flight f

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- **Bid-price heuristic** in network revenue management
- Broader principle of how to **price "products" through resource usage/cost**

Discrete Optimization

Today, we consider optimization problems with discrete variables:

$$
\begin{aligned}
\min\ & c^T x + d^T y \\
& Ax + By = b \\
& x, y \geq 0 \\
& x \text{ integer}
\end{aligned}
$$

This is called a **mixed integer programming** (MIP) problem

Without continuous variables y, it is called an **integer program** (IP)

If instead of $x \in \mathbb{Z}^n$ we have $x \in \{0,1\}^n$: **binary optimization** problem

Very powerful modeling paradigm

Example: Knapsack

- n items
- Item j has weight w_j and reward r_j
- Have a bound K on the weight that can be carried in the knapsack
- Want to select items to maximize the total value

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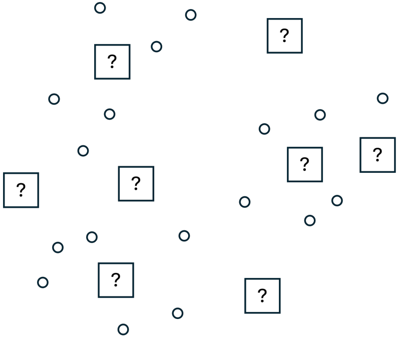
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} r_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} w_j x_j \leq K \\
& x_j \in \{0, 1\}, \quad j = 1, \ldots, n.
\end{aligned}
$$

Example: Facility Location

- n potential locations to open facilities
- Cost c_j for opening a facility at location j
- m clients who need service
- Cost d_{ij} for serving client i from facility j
- Smallest cost for opening facilities while serving all clients



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\min\ & \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_{ij} \\
& \sum_{j=1}^{n} x_{ij} = 1, \quad \forall\, i \\
& x_{ij} \leq y_j, \qquad \forall i,\ \forall j \\
& x_{ij}, y_j \in \{0, 1\}
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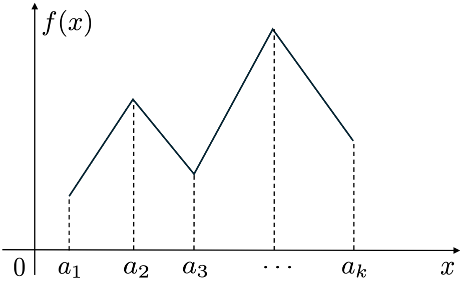
Which formulation is "better"?

Example: Piecewise Linear Cost



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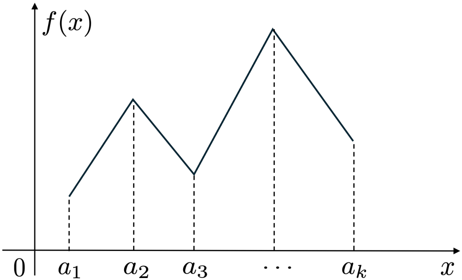

- Idea: $x = \sum_{i=1}^{k} \lambda_i a_i$
- Cost: $\sum_{i=1}^{k} \lambda_i f(a_i)$
- How to impose adjacency?

$$x = \lambda_i a_i + \lambda_{i+1} a_{i+1}$$

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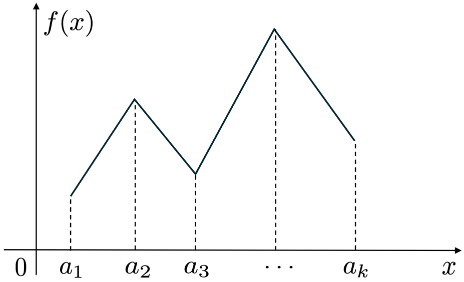

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$$y_j = 1 \ \Rightarrow \ \lambda_i = 0 \text{ for } i \notin \{j, j+1\}$$

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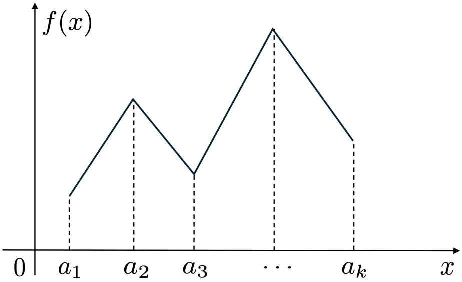

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$$\begin{aligned}
\textstyle\sum_{i=1}^{k} \lambda_i &= 1,\\
\lambda_1 &\leq y_1,\\
\lambda_i &\leq y_{i-1} + y_i, \; i = 2, \ldots, k-1,\\
\lambda_k &\leq y_{k-1},\\
\textstyle\sum_{i=1}^{k-1} y_i &= 1,\\
\lambda_i &\geq 0,\\
y_i &\in \{0, 1\}, \; \forall i.
\end{aligned}$$

Example: Matching Problems

- Set U of jobs/tasks to complete; set V of persons available to work
- Each task assigned to at most one person; a person can only complete some tasks
- Reward w_{ij} if task $i \in U$ completed by person $j \in V$

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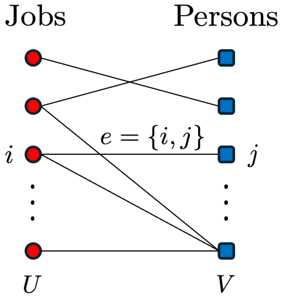

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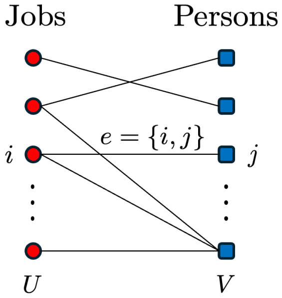

$x_e \in \{0, 1\}$: whether edge selected

$$\begin{aligned} \text{maximize} \quad & \sum_{e \in E} w_e x_e \\ & \sum_{e \in \delta(i)} x_e \leq 1, \quad \forall\, i \in N, \\ & x_e \in \{0, 1\}, \end{aligned}$$

$\delta(i) := \{j : \{i, j\} \in \mathcal{E}\}$: all neighbors of i

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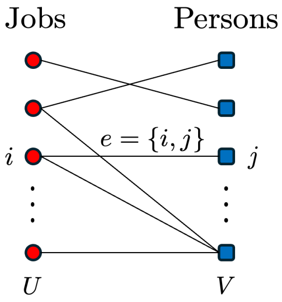

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Many variations: minimize cost, require jobs completed, perfect matching, ...

Example: Minimum Spanning Tree

- Given an undirected graph $G = (\mathcal{N}, \mathcal{E})$; $|\mathcal{N}| = n$, $|\mathcal{E}| = m$
- Edge $e \in \mathcal{E}$ has associated cost c_e
- Find **minimum spanning tree** (MST)
  (subset of edges that connect all nodes in $\mathcal{N}$ at minimum cost)

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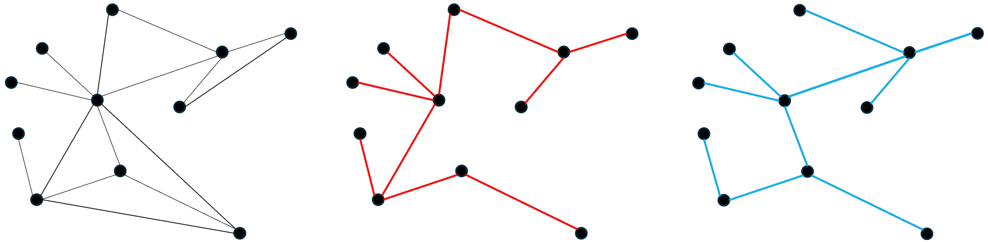

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& x_e \in \{0, 1\} \\
\textbf{(Connectivity)} \quad & \sum_{e \in \mathcal{E}} x_e = n - 1 \\
\textbf{(Cutset)} \quad & \sum_{e \in \delta(S)} x_e \geq 1, \; S \subset \mathcal{N}, S \neq \emptyset
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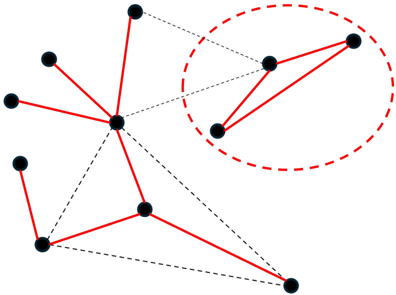

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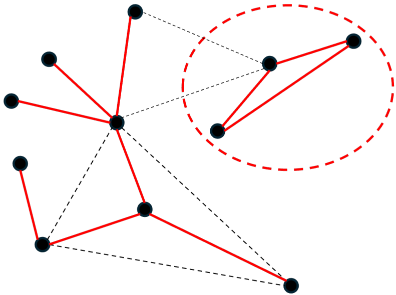
... or ...

(Subtour-elimination) $$\sum_{e \in \mathcal{E}(S)} x_e \leq |S| - 1, \;\; S \subset \mathcal{N}, S \neq \emptyset$$



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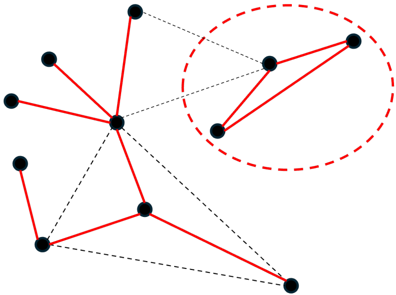

Again exponentially-sized formulations! *Any preference between them?*

Example: Traveling Salesperson Problem (TSP)

- Given an undirected graph $G = (\mathcal{N}, \mathcal{E})$; $|\mathcal{N}| = n$, $|\mathcal{E}| = m$
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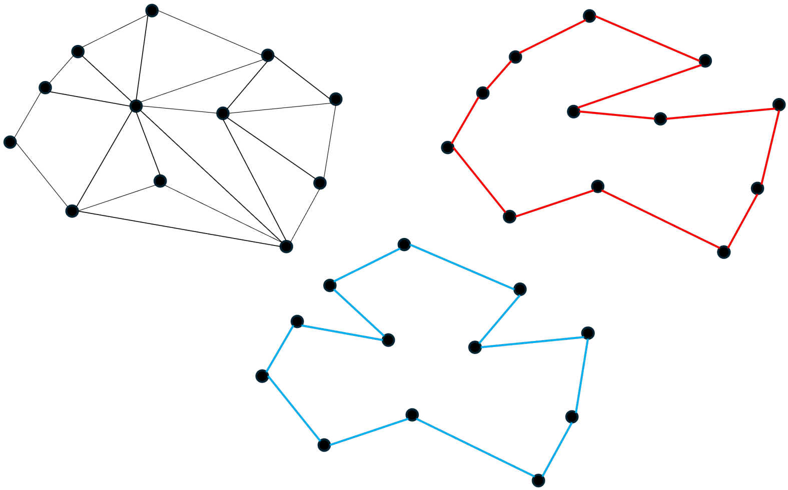

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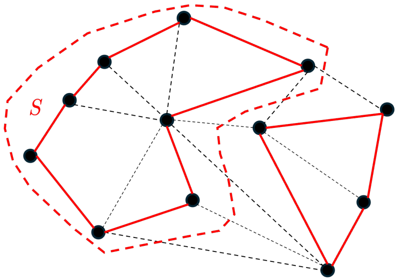

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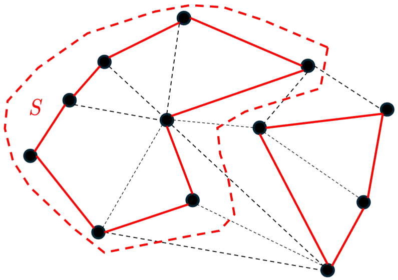

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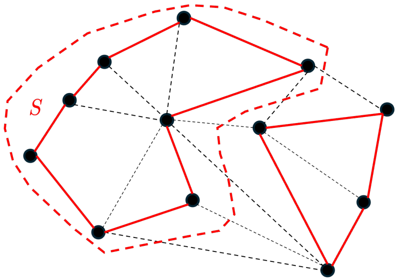

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- $x, p \in \mathbb{Z} \Rightarrow (\mathscr{P})$ infeasible, $(\mathscr{D})$ has optimal value 0.

  Strong duality does not hold in IPs

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Unfortunately, (M)IPs are **significantly harder** than LPs

Theorem

Given a matrix $A \in \mathbb{Q}^{m \times n}$ *and a vector* $b \in \mathbb{Q}^m$*, the problem: "does* $Ax \leq b$ *have an integral solution* x*" is* **NP-complete**.

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- IP "feasibility problem" is already in the hardest class of problems in NP
- Despite this, substantial body of theory and scalable algorithms exist for IPs
- We will focus on optimization problems with **rational entries**: $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$ (in fact, often **integer**)
- We assume that the **feasible set is bounded**

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Same question as in LP: *how can we find a good lower bound?*

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is obtained by replacing $x \in \{0,1\}^{n_1}$ with $x \in [0,1]^{n_1}$ and $y \in \mathbb{Z}^{n_2}$ with $y \in \mathbb{R}^{n_2}$.

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Key Q: How good is this bound?

LP Relaxation for Facility Location IP

Recall the **two** formulations of the Facility Location Problem

(FL)

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, m$$

$$x_{ij} \leq y_j, \quad i = 1, \ldots, m, \; j = 1, \ldots, n$$

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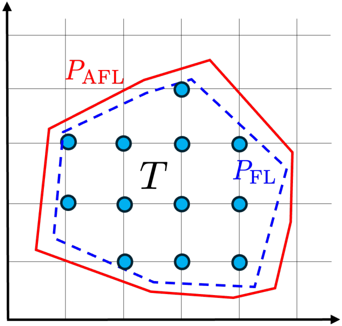
- $P_{\text{FL}} \subseteq P_{\text{AFL}}$ and can have **strict** inclusion
- **(FL) provides better lower bound than (AFL)**
- **Same IP feasible set, different LP feasible set!**



LP Relaxation for Minimum Spanning Tree Problem

(Cutset MST)

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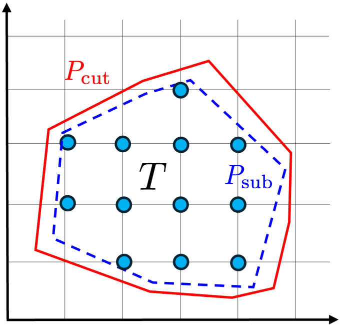
- $P_{\textbf{cut}}, P_{\textbf{sub}}$: feasible sets for LP relaxations
- $P_{\text{sub}} \subseteq P_{\text{cut}}$ and can have **strict** inclusion
  (Proof in the notes)
- **(SUB) provides better lower bound than (CUT)**
- **Same IP feasible set, different LP feasible set!**



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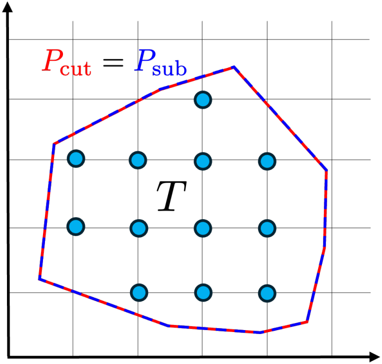

Strength of IP Formulation

- Different formulations of the same IP can result in **different LP relaxations**

- *What is an "ideal" formulation?*

Strength of IP Formulation

- T : all feasible points to an IP and $\text{conv}(T)$ is their convex hull
  - T finite because we assumed bounded feasible set
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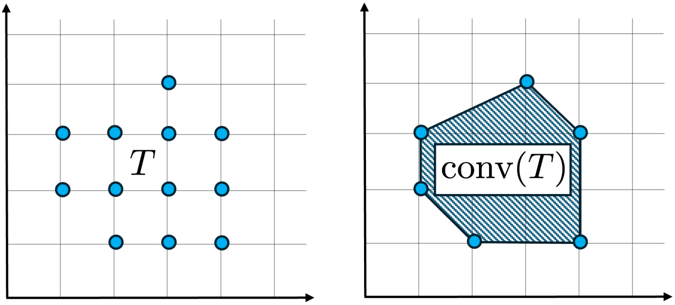

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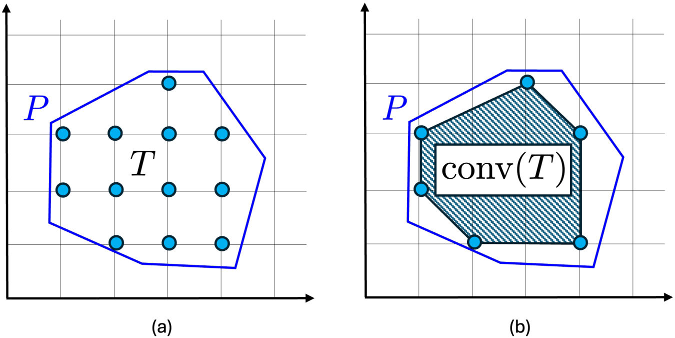

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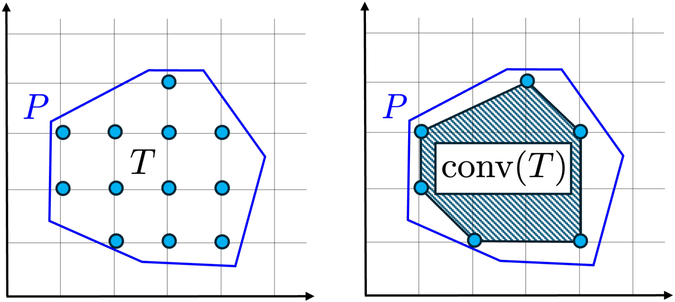

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Key take-aways:

- Quality of IP formulation : how closely its LP relaxation approximates $\text{conv}(T)$
- Formulation A is better than formulation B for some IP if $P_A \subset P_B$
- **Constraints** play a more subtle role in IPs than in LPs
  - Adding valid constraints for T that cut off fractional points from P is very useful!
  - More constraints not necessarily worse in IP!