Lecture 10 : Duality in Convex Optimization

October 30, 2024

Today's Agenda: Convex Duality

Primal Problem

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\begin{aligned}
(\mathscr{P})\ \text{minimize}_x \quad & f_0(x) \\
& f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& x \in X.
\end{aligned} \tag{1}
$$

- Convex domain $X \subseteq \mathbb{R}^n$
- Every function $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$ (real-valued), **convex**
- Equality constraints $Ax = b$ can be included in X

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- Many developments deal with the "interior" of X

Definition : Interior

The **interior** of a set X is the set of all points $x \in X$ so that:

$$\exists\, r > 0 \ :\ B(x, r) := \{y : \|y - x\| \leq r\} \subseteq X$$

Must talk about the interior even if X is not full-dimensional ...

Relative Interior

- **Recall**: **Affine hull** of X is $\text{aff}(X) := \left\{\theta_1 x_1 + \cdots + \theta_k x_k \ : \ x_i \in X, \sum_{i=1}^k \theta_i = 1\right\}$

Relative Interior

- **Recall**: **Affine hull** of X is $\text{aff}(X) := \{\theta_1 x_1 + \cdots + \theta_k x_k \ : \ x_i \in X, \sum_{i=1}^{k} \theta_i = 1\}$

Definition Relative Interior

The **relative interior** of a set X is:

$$\text{rel int}(X) := \left\{x \in X \ : \ \exists\, r > 0 \text{ so that } B(x,r) \cap \text{aff}(X) \subseteq X\right\}. \tag{2}$$

What is the relative interior of the following sets?

- $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \in [0,1]^2\}$
- $\{(x,y) \in \mathbb{R}^2 \mid x + y = 1, x \geq 0, y \geq 0\}$
- $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Convex Duality

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- Equality constraints $Ax = b$ can be included in X

- Assume $\mathrm{rel\ int}(X) \neq \emptyset$
- Assume that $(\mathscr{P})$ has an optimal solution $x^\star$, optimal value $p^\star = f_0(x^\star)$

- **Core questions:**
  1. For x feasible for $(\mathscr{P})$, how to **quantify the optimality gap** $f_0(x) - p^\star$?
  2. How to certify that $x^\star$ is **optimal** in $(\mathscr{P})$?

Convex Duality

Primal Problem

$$(\mathscr{P}) \ \text{minimize}_x \quad f_0(x)$$
$$f_i(x) \leq 0, \quad i = 1, \ldots, m$$
$$x \in X.$$

- To construct **lower bounds** for $(\mathscr{P})$, define the **Lagrangian function**: for $\lambda \geq 0$,

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x)$$

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- By construction, $\mathcal{L}(x, \lambda) \leq f_0(x)$ for any x feasible in $(\mathscr{P})$

Convex Duality

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- For a lower bound on $p^\star$, minimize $\mathcal{L}(x, \lambda)$ over $x \in X$

$$g(\lambda) := \inf_{x \in X} \mathcal{L}(x, \lambda).$$

Dual Problem

$$(\mathscr{D}) \quad \sup_{\lambda \ge 0} g(\lambda).$$

Q: Is the dual $(\mathscr{D})$ a convex optimization problem?

Convex Duality

Primal Problem

$$(\mathscr{P}) \ \text{minimize}_x \quad \begin{array}{ll} f_0(x) & \\ f_i(x) \leq 0, & i = 1, \ldots, m \\ x \in X. & \end{array}$$

- To construct **lower bounds** for $(\mathscr{P})$, define the **Lagrangian function**: for $\lambda \geq 0$,

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Dual Problem

$$(\mathscr{D}) \quad \sup_{\lambda \geq 0} g(\lambda).$$

Q: Is the dual $(\mathscr{D})$ a convex optimization problem? Yes, even if $(\mathscr{P})$ isn't!

Geometric Interpretation

Primal-Dual Pair

$$(\mathscr{P}) \; p^\star := \inf_{x \in X} \; f_0(x) \qquad\qquad (\mathscr{D}) \quad d^\star := \sup_{\lambda \geq 0} g(\lambda)$$

$$f_i(x) \leq 0, \; i = 1, \ldots, m$$

- Suppose $(\mathscr{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u, t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n, \; t = f_0(x), \; u = f_1(x)\}$

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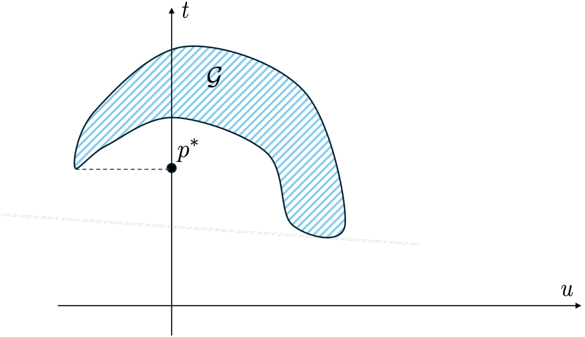

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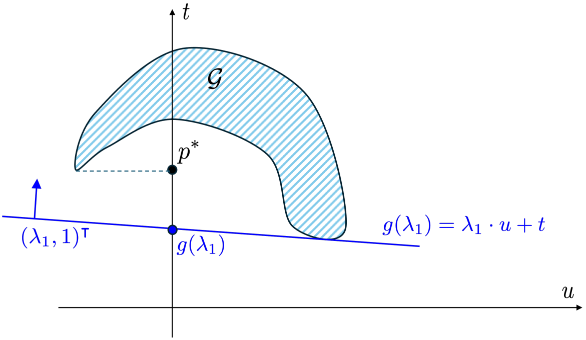

- Given $\lambda \geq 0$, to find $g(\lambda)$ we must minimize $t + \lambda \cdot u$ over $(u, t) \in \mathcal{G}$

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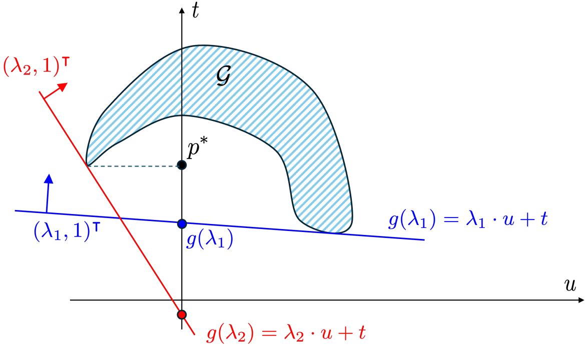

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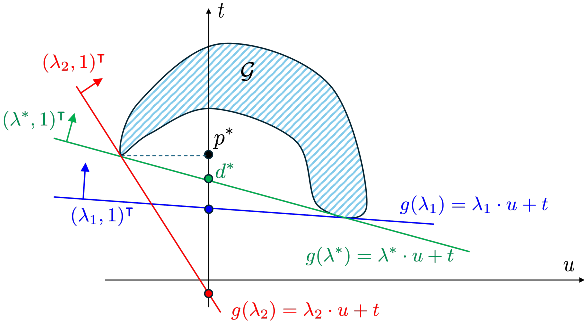

- Here, strong duality does not hold: $d^\star < p^\star$. But the set $\mathcal{G}$ is not convex!

Strong Duality in Convex Optimization?

Strong Duality in Convex Optimization?

Non-zero duality gap

Consider the example:

$$\underset{(x,y)\in X}{\text{minimize}}\ e^{-x}$$

$$x^2/y \leq 0$$

with domain $X = \{(x,y) \mid y \geq 1\}$.

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- *What are $p^\star, \mathcal{L}$, g, $d^\star$?*

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- *What are $p^\star, \mathcal{L}$, g, $d^\star$?*

$$p^\star = 1, \quad L(x,y,\lambda) = e^{-x} + \lambda x^2/y$$

$$g(\lambda) = \inf_{x,y\geq 1}\left(e^{-x} + \lambda\frac{x^2}{y}\right) = \begin{cases} 0 & \lambda \geq 0, \\ -\infty & \lambda < 0, \end{cases}$$

- We can write the dual problem as $d^\star = \max_{\lambda\geq 0} 0$, with optimal value $d^\star = 0$

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- We can write the dual problem as $d^\star = \max_{\lambda\geq 0} 0$, with optimal value $d^\star = 0$
- The optimal duality gap is $p^\star - d^\star = 1$
- Primal and dual both have finite optimal value, but a gap exists!
- Moreover, examples exist where $(\mathscr{D})$ does not achieve its optimal value... (notes)

Conditions Leading to Strong Duality

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Slater Condition

The functions $f_1, \ldots, f_m : X \subseteq \mathbb{R}^n \to \mathbb{R}$ satisfy **the Slater condition on** X if there exists $x \in \mathrm{rel\ int}(X)$ such that

$$f_j(x) < 0, \quad j = 1, \ldots, m.$$

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- A point x that is **strictly feasible**
- Condition simpler if some f_i are affine: only require $f_i(x) < 0$ for the **non-linear** f_i

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

Geometric intuition for proof:

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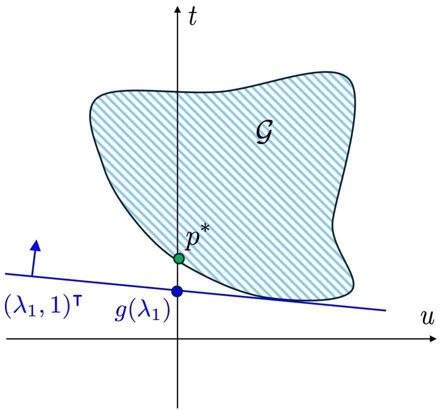
Geometric intuition for proof:

- Recall case with $m = 1$ and $\mathcal{G} := \{(u, t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



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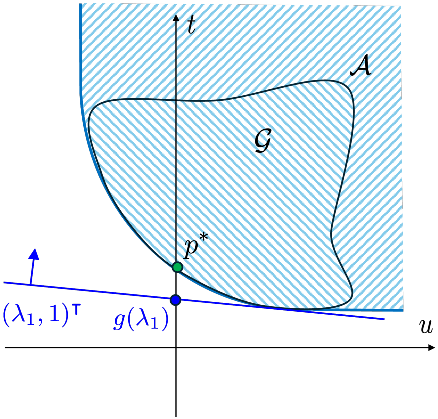
Geometric intuition for proof:

- Nothing changes if we replace $\mathcal{G}$ with $\mathcal{A} = \mathcal{G} + \mathbb{R}^2_+$, which is a **convex set**



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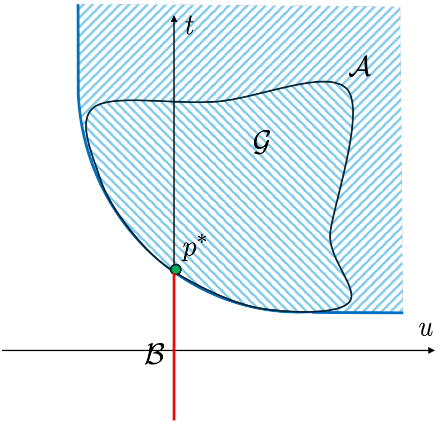
- Define another convex set $\mathcal{B}$ with $\mathcal{A} \cap \mathcal{B} = \emptyset$



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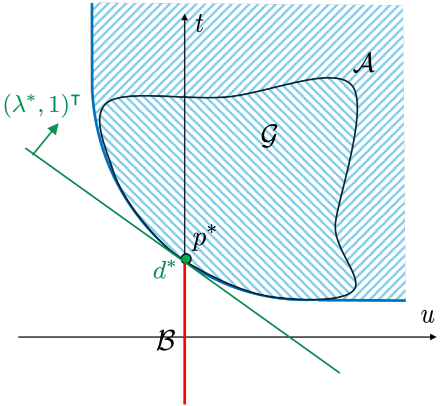
Geometric intuition for proof:

- The Separating Hyperplane Theorem will give us the optimal $\lambda^\star$ and $p^\star = d^\star$



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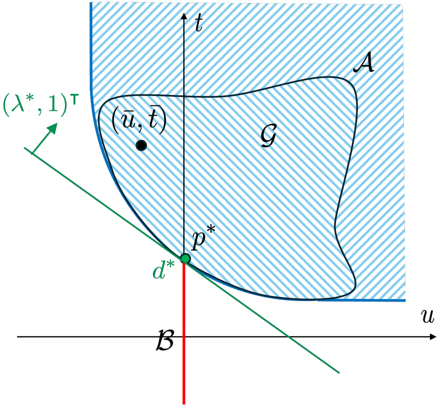
Geometric intuition for proof:

- The Slater point will guarantee that the hyperplane is not vertical



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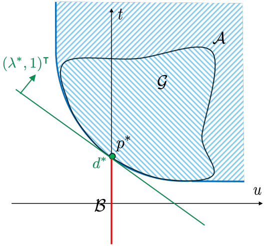
- Define the set

$$\mathcal{A} = \{(u,t) \in \mathbb{R}^m \times \mathbb{R} : \exists x \in X, \\ t \geq f_0(x), u_i \geq f_i(x),\ i = 1, \ldots, m\}.$$

- $\mathcal{A}$ is convex. *Why?*



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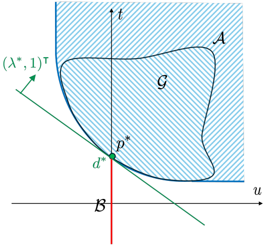
- $\mathcal{A}$ is convex. *Why?*
- Define the convex set $\mathcal{B} = \{(0, s) \in \mathbb{R}^m \times \mathbb{R} \mid s < p^\star\}$
- Claim: $\mathcal{A} \cap \mathcal{B} = \emptyset$. *Why?*



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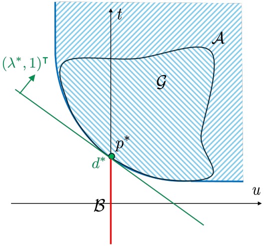
- Claim: $\mathcal{A} \cap \mathcal{B} = \emptyset$. *Why?*
- Separating Hyperplane Theorem:

$$\exists\, (\lambda, \mu) \in \mathbb{R}^{m+1},\ \ b \in \mathbb{R} : \begin{cases} (\lambda, \mu) \neq 0, \\ \lambda^\intercal u + \mu t \geq b, \forall\, (u,t) \in A \\ \lambda^\intercal u + \mu t \leq b, \forall\, (u,t) \in B. \end{cases}$$



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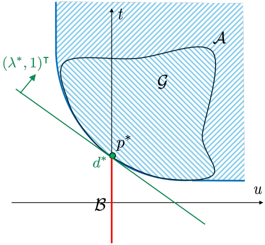

- (2) implies $\lambda \geq 0$ and $\mu \geq 0$.
  Otherwise, $\inf_{(u,t)\in\mathcal{A}}(\lambda^\intercal u + \mu t) = -\infty$ so $\ngeq b$ (*Why?*)

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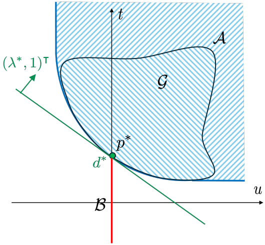

- (2) implies $\lambda \geq 0$ and $\mu \geq 0$.
  Otherwise, $\inf_{(u,t)\in\mathcal{A}}(\lambda^\intercal u + \mu t) = -\infty$ so $\not\geq b$ (*Why?*)
- (3) simplifies to $\mu t \leq b$ for all $t < p^\star$, so $\mu p^\star \leq b$.
- We found $\lambda \geq 0, \mu \geq 0$:

$$(4)\;\mathcal{L}(x,\lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star, \;\forall\, x \in X$$

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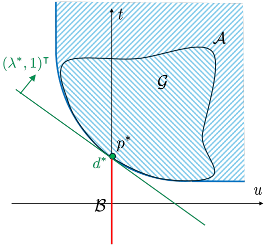
- **Case 1.** $\mu > 0$ (non-vertical hyper-plane)



Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

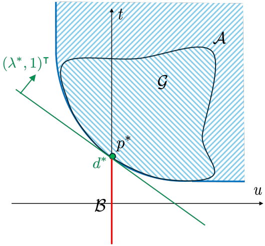
Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

- Separating Hyperplane Theorem:

$$\exists\,(\lambda,\mu) \in \mathbb{R}^{m+1},\quad b \in \mathbb{R} : \begin{cases} (1) & (\lambda,\mu) \neq 0, \\ (2) & \lambda^\intercal u + \mu t \geq b, \forall\,(u,t) \in A \\ (3) & \lambda^\intercal u + \mu t \leq b, \forall\,(u,t) \in B. \end{cases}$$

- We found $\lambda \geq 0, \mu \geq 0$:

$$(4)\ \mathcal{L}(x,\lambda) := \sum_{i=1}^{m} \lambda_i f_i(x) + \mu f_0(x) \geq b \geq \mu p^\star,\ \forall\, x \in X$$



- **Case 1.** $\mu > 0$ (non-vertical hyper-plane)
- Divide (4) by μ to get: $\mathcal{L}(x,\lambda/\mu) \geq p^\star$, $\forall\, x \in X$.
- This implies $g(\lambda/\mu) \geq p^\star$
- Weak duality: $g(\lambda/\mu) \leq p^\star$, so $g(\lambda/\mu) = p^\star$
- Strong duality holds and the dual optimum is attained

Strong Duality in Convex Optimization

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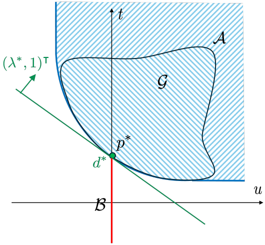
- (4) implies $\sum_{i=1}^{m} \lambda_i f_i(x) \geq 0, \forall\, x \in X$



Strong Duality in Convex Optimization

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$$\exists\, (\lambda, \mu) \in \mathbb{R}^{m+1}, \quad b \in \mathbb{R} : \begin{cases} (1) & (\lambda, \mu) \neq 0, \\ (2) & \lambda^\intercal u + \mu t \geq b, \forall\, (u,t) \in A \\ (3) & \lambda^\intercal u + \mu t \leq b, \forall\, (u,t) \in B. \end{cases}$$

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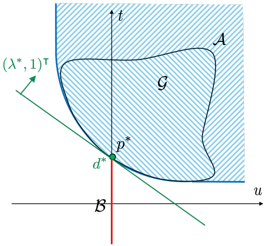

- **Case 2.** $\mu = 0$ (vertical hyperplane)
- (4) implies $\sum_{i=1}^{m} \lambda_i f_i(x) \geq 0,\ \forall\, x \in X$
- $\bar{x}$ satisfies Slater condition $\Rightarrow f_i(\bar{x}) < 0$ for $i = 1, \ldots, m$
- This together with $\lambda \geq 0 \Rightarrow \lambda = 0$
- Contradicts (1) that $(\lambda, \mu) \neq 0$.

Explicit Equality Constraints

- In applications, useful to make the **equality constraints explicit**:

$$\begin{aligned} \text{minimize}_{x \in \mathcal{X}} \;& f_0(x) \\ \text{subject to } & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & Ax = b. \end{aligned}$$

where $f_i, i = 0, \ldots, m$ are convex and $A \in \mathbb{R}^{p \times n}$ has rank p.

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$$\mathcal{L}(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \nu^\mathsf{T}(Ax - b),$$

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- With $g(\lambda, \nu) := \inf_{x \in X} \mathcal{L}(x, \lambda, \nu)$, the dual problem becomes:

$$\begin{aligned} \text{maximize } & g(\lambda, \nu) \\ \text{subject to } & \lambda \geq 0. \end{aligned}$$

  No sign constraints on ν!

Nonlinear Farkas Lemma

Proposition (Nonlinear Farkas Lemma)

Let $X \subset \mathbb{R}^n$ be convex, let $f_0, f_1, \ldots, f_m$ be real-valued convex functions on X, and assume $f_1, \ldots, f_m$ satisfy the Slater condition on X.

Then, the following system of inequalities has a solution

$$\exists\, x \ : \ f_0(x) < z, \quad f_j(x) \leq 0,\ j = 1, \ldots, m, \quad x \in X,$$

if and only if the following system has no solution:

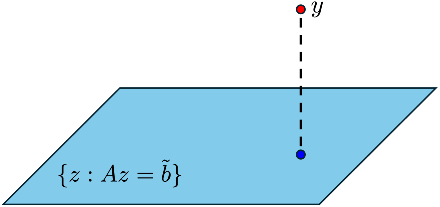
$$\exists\, \lambda \ : \ \inf_{x \in X} \left[f(x) + \sum_{j=1}^{m} \lambda_j f_j(x) \right] \geq z, \ \ \lambda_j \geq 0,\ j = 1, \ldots, m.$$

- Mirrors arguments used in strong duality proof

Minimum Euclidean Distance Problem

- Given $y \in \mathbb{R}^n$ and affine set $\{z : Az = \tilde{b}\}$
- $A \in \mathbb{R}^{p \times n}$, $\tilde{b} \in \mathbb{R}^p$ has rank p

$$\min_z \left\{ \|z - y\|_2^2 \ : \ Az = \tilde{b} \right\}$$



- Change of variables $x := z - y$ and with $b := \tilde{b} - Ay$,

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- *What is the optimal value $p^\star$?*

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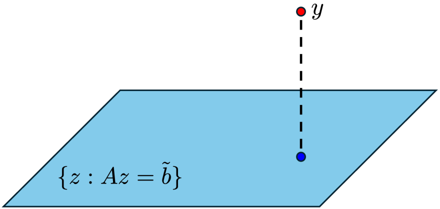

- Change of variables $x := z - y$ and with $b := \tilde{b} - Ay$,

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- Lagrangian $L(x, \nu) = x^\mathsf{T} x + \nu^\mathsf{T}(Ax - b)$: convex quadratic function of x
- Dual objective: $g(\nu) = \inf_x L(x, \nu)$. Can find via:

$$\nabla_x L(x, \nu) = 2x + A^\mathsf{T}\nu = 0 \quad \Leftrightarrow \quad x = -\frac{1}{2}A^\mathsf{T}\nu$$

- $g(\nu) = L\left(-\frac{1}{2}A^\mathsf{T}\nu, \nu\right) = -\frac{1}{4}\nu^\mathsf{T} A A^\mathsf{T} \nu - b^\mathsf{T}\nu$
- Primal trivially satisfies Slater condition (if it is feasible) so $p^\star = d^\star$
- To find $d^\star$:

$$\nabla_\nu g(\nu) = 0 \quad \Leftrightarrow \quad -\frac{1}{2}AA^\mathsf{T}\nu = b.$$

- AA^T is invertible, so $\nu^\star = -2(AA^\mathsf{T})^{-1}b$, $p^\star = d^\star = g(\nu^\star) = b^\mathsf{T}(AA^T)^{-1}b$
- $x^\star = -\frac{1}{2}A^\mathsf{T}\nu^\star = A^\mathsf{T}(AA^\mathsf{T})^{-1}b$

Quadratic Programs - Preliminaries

Unconstrained Quadratic Program

For $Q = Q^\intercal$, consider the following unconstrained problem:

$$\min\ f(x) := \frac{1}{2}x^\intercal P x + q^\intercal x$$

- *What is the optimal value $p^\star$?*

Quadratic Programs - Preliminaries

Unconstrained Quadratic Program

For $Q = Q^\intercal$, consider the following unconstrained problem:

$$\min\ f(x) := \frac{1}{2}x^\intercal P x + q^\intercal x$$

- *What is the optimal value $p^\star$?*

$$\nabla_x f(x) = 0 \quad \Leftrightarrow \quad Px = -q$$

$$p^\star = \begin{cases} -\frac{1}{2}q^\intercal P^\dagger q & \text{if } P \succeq 0 \text{ and } q \in \mathcal{R}(P) \\ -\infty & \text{otherwise.} \end{cases}$$

- $P^\dagger$ is the (Moore-Penrose) pseudo-inverse of P
- For A with singular value decomposition $A = U\Sigma V^\intercal$, $A^\dagger := V\Sigma^{-1}U^\intercal$
- Equals $(A^\intercal A)^{-1}A^\intercal$ if $\text{rank}(A) = n$ and $A^\intercal(AA^\intercal)^{-1}$ if $\text{rank}(A) = m$