Optimization Under Uncertainty

(but really, just Robust Optimization)

Lecture 18

December 2, 2024

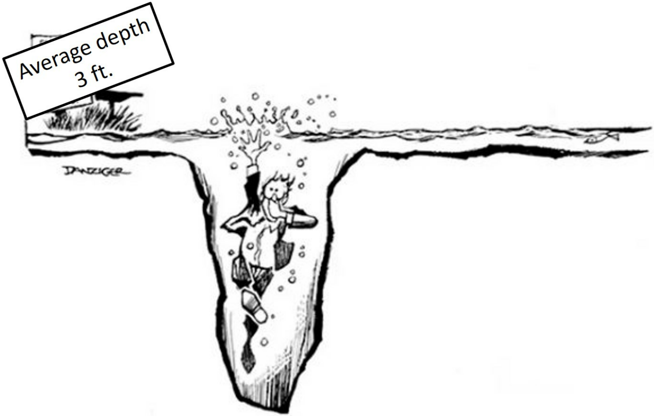
Quick Announcements

- Homework 5 due on Tuesday (Dec 3)
- Office Hours this week - extended schedule (Ed Announcement coming up)
- Final exam topics
- Any questions?

Outline for Today

1. Introduction
   - Some Motivating Examples
   - A History Detour
   - Pros and Cons of Probabilistic Models
2. Robust Optimization
   - Basic Premises
   - Modeling with Basic Uncertainty Sets
   - Reformulating and Solving Robust Models
   - Extensions
   - Some Applications
   - Calibrating Uncertainty Sets
   - Distributionally Robust Optimization
   - Connections with Other Areas
3. Dynamic Robust Optimization
   - Properly Writing a Robust DP
   - An Inventory Example
   - Tractable Approximations with Decision Rules
   - Some Practical Issues
   - Bellman Optimality
   - An Application in Monitoring

Optimization based on nominal values can lead to severe issues...



Taken from *"Flaw of averages"* Sam Savage (2009, 2012)

How Robust Are Optimal Solutions? (Ben-Tal & Nemirovski)

- Consider a **real-world scheduling problem** problem (PILOT4) in NETLIB Library
  - One of the constraints is the following linear constraint $\bar{\mathbf{a}}^\top \mathbf{x} \geqslant b$:

$$
\begin{aligned}
&-15.79081 \cdot x_{826} - \textcolor{blue}{\mathit{8.598819}} \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}
$$

  - Coefficients like *8.598819* are estimated and potentially inaccurate

How Robust Are Optimal Solutions? (Ben-Tal & Nemirovski)

- Consider a **real-world scheduling problem** problem (PILOT4) in NETLIB Library
  - One of the constraints is the following linear constraint $\bar{\mathbf{a}}^\top \mathbf{x} \geqslant b$:

$$
\begin{aligned}
&-15.79081 \cdot x_{826} - \mathit{8.598819} \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}
$$

  - Coefficients like *8.598819* are estimated and potentially inaccurate

- What if these coefficients are just 0.1% inaccurate?
  - i.e., suppose the true $\mathbf{a}$ is not $\bar{\mathbf{a}}$, but $|a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|$?

- Will the optimal solution to the problem still be feasible?

- How can we test?

How Robust Are Optimal Solutions? (Ben-Tal & Nemirovski)

- Original constraint: $\bar{\mathbf{a}}^\top \boldsymbol{x} \geqslant b$, optimal solution $\boldsymbol{x}^\star$
- Suppose true $\mathbf{a}$ satisfies $|a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|, \forall\, i$

- How to determine if the constraint is violated?

How Robust Are Optimal Solutions? (Ben-Tal & Nemirovski)

- Original constraint: $\bar{\boldsymbol{a}}^\mathsf{T}\boldsymbol{x} \geqslant b$, optimal solution $\boldsymbol{x}^\star$
- Suppose true $\boldsymbol{a}$ satisfies $|a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|, \forall\, i$

- How to determine if the constraint is violated?

$$\begin{aligned} \min_{\boldsymbol{a}} \;& \boldsymbol{a}^\mathsf{T}\boldsymbol{x}^\star - b \\ \text{s.t. } \;& |a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|, \forall\, i \end{aligned}$$

  - For PILOT4, this comes to $-128.8 \approx -4.5b$, so 450% violation!

How Robust Are Optimal Solutions? (Ben-Tal & Nemirovski)

- Original constraint: $\bar{\mathbf{a}}^\top \mathbf{x} \geqslant b$, optimal solution $\mathbf{x}^\star$
- Suppose true $\mathbf{a}$ satisfies $|a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|$, $\forall\, i$

- How to determine if the constraint is violated?

$$\begin{aligned} \min_{\mathbf{a}}\ & \mathbf{a}^\top \mathbf{x}^\star - b \\ \text{s.t.}\ & |a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|, \ \forall\, i \end{aligned}$$

  - For PILOT4, this comes to $-128.8 \approx -4.5b$, so 450% violation!

- OK, but perhaps we're too conservative?
  - Suppose $a_i = \bar{a}_i + \varepsilon_i|\bar{a}_i|$, where $\varepsilon_i \sim \text{Uniform}[-0.001, 0.001]$
  - Using Monte-Carlo simulation with 1,000 samples:
    - $\mathbb{P}(\text{infeasible}) = 50\%$, $\mathbb{P}(\text{violation} > 150\%) = 18\%$, $\mathbb{E}[\text{violation}] = 125\%$

How Robust Are Optimal Solutions? (Ben-Tal & Nemirovski)

- Original constraint: $\bar{a}^\top x \geqslant b$, optimal solution $x^\star$
- Suppose true a satisfies $|a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|, \forall\, i$

- How to determine if the constraint is violated?

$$\min_{a}\; a^\top x^\star - b$$
$$\text{s.t.}\;\; |a_i - \bar{a}_i| \leqslant 0.001|\bar{a}_i|, \forall\, i$$

  - For PILOT4, this comes to $-128.8 \approx -4.5b$, so 450% violation!

- OK, but perhaps we're too conservative?
  - Suppose $a_i = \bar{a}_i + \varepsilon_i|\bar{a}_i|$, where $\varepsilon_i \sim \text{Uniform}[-0.001, 0.001]$
  - Using Monte-Carlo simulation with 1,000 samples:
    - $\mathbb{P}(\text{infeasible}) = 50\%$, $\mathbb{P}(\text{violation} > 150\%) = 18\%$, $\mathbb{E}[\text{violation}] = 125\%$

- Disturbing that nominal solutions are likely highly infeasible
- Turns out to be the case for many **NETLIB** problems
- We should **capture uncertainty more explicitly** apriori!

Decisions Under Uncertainty

- Decision Maker (DM) must chose x, without knowing z

- DM incurs a **cost** $C(x, z)$

Decisions Under Uncertainty

- Decision Maker (DM) must chose x, without knowing z

- DM incurs a **cost** $C(x, z)$

- How to model z? How to properly formalize the decision problem?

Decisions Under Uncertainty

- Decision Maker (DM) must chose x, without knowing z

- DM incurs a **cost** $C(x, z)$

- How to model z? How to properly formalize the decision problem?

- "Standard" probabilistic model:
  - There is a unique probability distribution $\mathbb{P}$ for z
  - DM considers an objective: $\min_x \ \mathbb{E}_{z \sim \mathbb{P}}\big[C(x, z)\big]$

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_x \ \mathbb{E}_{z\sim\mathbb{P}}\left[C(x, z)\right]$

- What if there are constraints?

$$f_i(x, z) \geqslant 0, \ \forall i \in I$$

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_x \; \mathbb{E}_{z\sim\mathbb{P}}\big[C(x,z)\big]$

- What if there are constraints?

$$f_i\,(x, z) \geqslant 0, \;\forall\, i \in I$$

- Need to be a bit more precise in which **sense** we want to satisfy them!
  - expectation constraint: $\mathbb{E}_{\mathbb{P}}[\,f_i\big(x, z\big)] \geqslant 0, \;\forall\, i$
  - chance constraint:
    individual: $\mathbb{P}[\,f_i\big(x, z\big) \geqslant 0\,] \geqslant 1 - \epsilon, \;\forall\, i$
    joint: $\mathbb{P}[\,f_i\big(x, z\big) \geqslant 0, \;\forall\, i] \geqslant 1 - \epsilon$
  - robust (a.s.) constraint: $F\big(x, z\big) \geqslant 0, \forall\, z$

- Which of these are "easy" to check / enforce?

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_x \; \mathbb{E}_{z\sim\mathbb{P}}\big[C(x, z)\big]$

- What if there are constraints?

$$f_i\,(x, z) \geqslant 0,\ \forall\, i \in I$$

- Need to be a bit more precise in which **sense** we want to satisfy them!
  - expectation constraint: $\mathbb{E}_{\mathbb{P}}[\,f_i(x, z)] \geqslant 0,\ \forall\, i$
  - chance constraint:

    individual: $\mathbb{P}[\,f_i(x, z) \geqslant 0\,] \geqslant 1 - \epsilon,\ \forall\, i$

    joint: $\mathbb{P}[\,f_i(x, z) \geqslant 0,\ \forall\, i] \geqslant 1 - \epsilon$
  - robust (a.s.) constraint: $F(x, z) \geqslant 0, \forall\, z$

- Which of these are "easy" to check / enforce?

- Even if f is "well-behaved," may need some assumptions on $\mathbb{P}$
  - e.g., f convex in x, concave in z
  - log-concave density for chance constraints
  - convex support

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_{x} \ \mathbb{E}_{z \sim \mathbb{P}}\big[C(x, z)\big]$

- Where is $\mathbb{P}$ coming from?
- When is this reasonable?
- What if $\mathbb{P}$ is not the actual distribution?
- What if $\mathbb{P}$ is not exogenous?

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_x \ \mathbb{E}_{z\sim\mathbb{P}}\big[C(x, z)\big]$

- Where is $\mathbb{P}$ coming from?
- When is this reasonable?
- What if $\mathbb{P}$ is not the actual distribution?
- What if $\mathbb{P}$ is not exogenous?

- Perhaps we have historical samples $z_1, \ldots, z_N$
- Use empirical distribution $\mathbb{P} = \sum_{i=1}^{N} \frac{1}{N}\delta(z_i)$?
- Future like the past...
- ...

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_x \; \mathbb{E}_{z\sim\mathbb{P}}\big[C(x, z)\big]$

- What if there are constraints?
  $f_i\,(x, z) \geqslant 0,\ \forall\, i \in I$
- Where is $\mathbb{P}$ coming from?
- When is this reasonable?
- What if $\mathbb{P}$ is not the actual distribution?
- What if $\mathbb{P}$ is not exogenous?

- **Very** popular modeling framework, but...

Classical Probabilistic Model: DM knows $\mathbb{P}$, solves $\min_x \; \mathbb{E}_{z\sim\mathbb{P}}\big[C(x, z)\big]$

- What if there are constraints?
  $f_i(x, z) \geqslant 0, \; \forall i \in I$
- Where is $\mathbb{P}$ coming from?
- When is this reasonable?
- What if $\mathbb{P}$ is not the actual distribution?
- What if $\mathbb{P}$ is not exogenous?

- **Very** popular modeling framework, but...
- Theory unable to analyze **complex, real-world** phenomena
  - poor data, changing environments (future $\neq$ past), many agents, ...
- Framework not geared towards **computing decisions**
  - Limited computational tractability, particularly in higher dimensions
- With $C = -u(\cdot)$ (u utility function), unclear if this is a good behavioral model

An Alternative Model of Uncertainty

- Let's admit **explicitly** that our model of reality is **incorrect**
- From **classical view**: *"we know distribution* $\mathbb{P}$ *for* $\boldsymbol{z}$*, and solve:* $\min_{\boldsymbol{x}} \mathbb{E}_{\mathbb{P}}\big[C(\boldsymbol{x}, \boldsymbol{z})\big]$*"*

  to **robust view**: *"we only know that* $\mathbb{P} \in \mathcal{P}$*, and solve:* $\min_{\boldsymbol{x}} \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[C(\boldsymbol{x}, \boldsymbol{z})\big]$*"*

An Alternative Model of Uncertainty

- Let's admit **explicitly** that our model of reality is **incorrect**
- From **classical view**: *"we know distribution* $\mathbb{P}$ *for* z*, and solve:* $\min_{x} \mathbb{E}_{\mathbb{P}}\big[C(x, z)\big]$*"*

  to **robust view**: *"we only know that* $\mathbb{P} \in \mathcal{P}$*, and solve:* $\min_{x} \max_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\big[C(x, z)\big]$*"*

Long history of **robust decision-making** and **model misspecification**:

- **Economics:**
  - Frank Knight (1921) - risk vs. Knightian uncertainty, Abraham Wald (1939), John von Neumann (1944) zero-sum games
  - Savage (1951): minimax regret, Scarf (1958): robust Newsvendor model
  - Schmeidler, Gilboa (1980s): axiomatic frameworks, Ben-Haim (1980s): info-gap theory
  - Hansen & Sargent (2008): *"Robustness"* - robust control in macroeconomics
  - Bergemann & Morris (2012): *"Robust mechanism design"* book, Carroll (2015), ...
- **Engineering and robust control:** Bertsekas (1970s), Doyle (1980s), etc.
- **Computer science:** complexity analysis; adversarial training (modern!)
- **Statistics:** M-estimators Huber (1981)
- **Operations Research:**
  - Early work by Soyster (1973), Libura (1980), Bard (1984), Kouvelis (1997)
  - **Robust Optimization**: Ben-Tal, Nemirovski, El-Ghaoui ('90s), Bertsimas, Sim ('00s)
  - Two books: Ben-Tal, El-Ghaoui, Nemirovski (2009), Bertsimas, den Hertog (2020)
  - Many tutorials!

An Alternative Model of Uncertainty

- Let's admit **explicitly** that our model of reality is **incorrect**
- From **classical view**: *"we know distribution* $\mathbb{P}$ *for* z*, and solve:* $\min_{x} \mathbb{E}_{\mathbb{P}}\left[C(x, z)\right]$*"*

  to **robust view**: *"we only know that* $\mathbb{P} \in \mathcal{P}$*, and solve:* $\min_{x} \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[C(x, z)\right]$*"*

Why robust optimization? (in my view)

1. Very sensible
2. Modest modeling requirements
3. Modest in its premise: *"always under-promises, and over-delivers"*
4. Tractable: quickly becoming "technology"
5. Very sensible results: can rationalize simple rules in complex problems

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$

- DM reformulates the original optimization problem as:

$$(P) \quad \begin{aligned} & \inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ & \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(P) \quad \begin{array}{l} \inf\limits_{x} \sup\limits_{z \in \mathcal{U}} \; C(x, z) \\ \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{array}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
   - Other options possible, based on notions of **regret**

- Conservative?

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(\mathrm{P}) \quad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ &\text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} C(x, z)$
   - Other options possible, based on notions of **regret**

- Conservative?
  - **Not necessarily!**
  - $\mathcal{U}$ directly trades off robustness and conservatism, and is ultimately a **modeling choice**

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$
(\mathrm{P}) \qquad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} \; C(x, z) \\ &\text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \; \forall i \in I \end{aligned}
$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
   - Other options possible, based on notions of **regret**

- Conservative?
  - **Not necessarily!**
  - $\mathcal{U}$ directly trades off robustness and conservatism, and is ultimately a **modeling choice**

- Is there a probabilistic interpretation?

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(P) \quad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ &\text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} C(x, z)$
   - Other options possible, based on notions of **regret**

- Conservative?
   - **Not necessarily!**
   - $\mathcal{U}$ directly trades off robustness and conservatism, and is ultimately a **modeling choice**

- Is there a probabilistic interpretation?
   - Objective $= \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{z \sim \mathbb{P}}[C(x, z)]$ where $\mathcal{P}$ is the set of all measures with support $\mathcal{U}$
   - So we are assuming that the only information about $\mathbb{P}$ is the support $\mathcal{U}$

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$
(P) \qquad \begin{array}{l} \inf\limits_{x} \sup\limits_{z \in \mathcal{U}} \; C(x, z) \\ \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{array}
$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z

What is the optimal value of the following robust LP?

$$
\begin{array}{ll} \min\limits_{x} \max\limits_{a \in \mathcal{U}} & -(x_1 + x_2) \\ \text{such that} & x_1 \leqslant a_1 \\ & x_2 \leqslant a_2 \\ & x_1 + x_2 \leqslant 1. \end{array} \qquad \text{where } \mathcal{U} = \{(a_1, a_2) \in [0,1]^2 : a_1 + a_2 \leqslant 1\}
$$

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(\mathrm{P}) \qquad \begin{aligned} & \inf_{x} \ \sup_{z \in \mathcal{U}} \ C(x, z) \\ & \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \ \forall i \in I \end{aligned}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \ C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z

What is the optimal value of the following robust LP?

$$\begin{aligned} \min_{x} \max_{a \in \mathcal{U}} \quad & -(x_1 + x_2) \\ \text{such that} \quad & x_1 \leqslant a_1 \\ & x_2 \leqslant a_2 \qquad\qquad \text{where } \mathcal{U} = \{(a_1, a_2) \in [0,1]^2 \ : \ a_1 + a_2 \leqslant 1\} \\ & x_1 + x_2 \leqslant 1. \end{aligned}$$

Optimal value 0. In RO, **each constraint must be satisfied separately, robustly.**

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(\mathrm{P}) \qquad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} \; C(x, z) \\ &\text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$
(\mathrm{P}) \qquad \begin{array}{l} \inf_{x} \sup_{z \in \mathcal{U}} \; C(x, z) \\ \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \; \forall i \in I \end{array}
$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem

$$
\boxed{f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}} \quad \Leftrightarrow \quad \boxed{\sup_{z \in \mathcal{U}} f_i(x, z) \leqslant 0}
$$

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(\mathrm{P}) \qquad \begin{aligned} & \inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ & \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem
4. Without loss, we can consider a problem where z only appears in constraints

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(\mathrm{P}) \qquad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ &\text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem
4. Without loss, we can consider a problem where z only appears in constraints

(P) is equivalent to the following problem:

$$\begin{aligned} &\inf_{x,t} \ t \\ &\text{s.t. } t \geqslant C(x, z), \forall z \in \mathcal{U} \\ &\qquad f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(P) \quad \begin{aligned} &\inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ &\text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem
4. Without loss, we can consider a problem where z only appears in constraints

   (P) is equivalent to the following problem:

$$\begin{aligned} &\inf_{x,t} \; t \\ &\text{s.t. } t \geqslant C(x, z), \forall z \in \mathcal{U} \\ &\qquad f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

   Many RO models are in this *epigraph reformulation*, and focus on constraints

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(\mathrm{P}) \qquad \begin{array}{l} \inf_{x} \sup_{z \in \mathcal{U}} \; C(x, z) \\ \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \; \forall i \in I \end{array}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} \; C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem
4. Without loss, we can consider a problem where z only appears in constraints
5. DM only responsible for objective and constraints when $z \in \mathcal{U}$
   - If $z \notin \mathcal{U}$ actually occurs, all bets are off
   - Can extend framework to ensure **gradual** degradation of performance: Globalized robust counterparts (Ben-Tal & Nemirovski)

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(P) \quad \begin{array}{l} \inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ \text{s.t. } f_i(x, z) \leq 0, \forall z \in \mathcal{U}, \forall i \in I \end{array}$$

Remarks.

1. Objective: worst-case performance $\sup_{z \in \mathcal{U}} C(x, z)$
2. Each constraint is "hard": must be satisfied *robustly*, for any realization of z
3. Each constraint can be re-written as an optimization problem
4. Without loss, we can consider a problem where z only appears in constraints
5. DM only responsible for objective and constraints when $z \in \mathcal{U}$
6. Robust model seems to lead to a **difficult** optimization problem
   - For any given x, checking constraints/solving the "adversary" problem may be tough
   - We must also solve our original problem of finding x!

"Classical" Robust Optimization (RO)

- Robust Optimization: the values of z belong to an **uncertainty set** $\mathcal{U}$
- DM reformulates the original optimization problem as:

$$(P) \quad \begin{aligned} & \inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ & \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

1. How to model $\mathcal{U}$
2. How to formulate and solve the **robust counterpart**
3. Why is this useful, in theory and in practice

Intuition for Some Basic Uncertainty Sets

- Recall PILOT4; how to build some "safety buffers" for constraint like #372:

$$
\begin{aligned}
&-15.79081 \cdot x_{826} - \mathit{8.598819} \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}
$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$
\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}
$$

Intuition for Some Basic Uncertainty Sets

- Recall PILOT4; how to build some "safety buffers" for constraint like #372:

$$
\begin{aligned}
&-15.79081 \cdot x_{826} - 8.598819 \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}
$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$
\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}
$$

- P is a known matrix; z is primitive uncertainty

- **Q:** Why this more general form?

  A: For modeling flexibility:
  - Suppose the same physical quantity (i.e., coefficient) appears in multiple constraints
  - Can capture "correlations", e.g., with a factor model

Intuition for Some Basic Uncertainty Sets

- Recall PILOT4; how to build some "safety buffers" for constraint like #372:

$$
\begin{aligned}
&-15.79081 \cdot x_{826} - 8.598819 \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}
$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$
\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}
$$

- How about a **box** uncertainty set? For some confidence level ρ:

$$
\mathcal{U}_{\text{box}} := \{z : -\rho \leqslant z_i \leqslant \rho\} = \{z : \|z\|_\infty \leqslant \rho\}
$$

"Too conservative?"

- In PILOT4, **robust** solution is within 1% of x^* for objective
- Recall that x^* would violate this constraint by 450%
- Sometimes not much is sacrificed for robustness!

Intuition for Some Basic Uncertainty Sets

- Recall PILOT4; how to build some "safety buffers" for constraint like #372:

$$
\begin{aligned}
&-15.79081 \cdot x_{826} - 8.598819 \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}
$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$
\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}
$$

- How about a **box** uncertainty set? For some confidence level ρ:

$$
\mathcal{U}_{\text{box}} := \{z : -\rho \leqslant z_i \leqslant \rho\} = \{z : \|z\|_\infty \leqslant \rho\}
$$

- Or maybe an **ellipsoid** would be less conservative:

$$
\mathcal{U}_{\text{ellipsoid}} := \{z : \|z\|_2 \leqslant \rho\}
$$

Intuition for Some Basic Uncertainty Sets

- Recall PILOT4; how to build some "safety buffers" for constraint like #372:

$$\begin{aligned}
&-15.79081 \cdot x_{826} - 8.598819 \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}$$

- How about a **box** uncertainty set? For some confidence level ρ:

$$\mathcal{U}_{\text{box}} := \{z : -\rho \leqslant z_i \leqslant \rho\} = \{z : \|z\|_\infty \leqslant \rho\}$$

- Or maybe an **ellipsoid** would be less conservative:

$$\mathcal{U}_{\text{ellipsoid}} := \{z : \|z\|_2 \leqslant \rho\}$$

- Or what if we gave "nature" a **budget** on how many coefficients it could change:

$$\mathcal{U}_{\text{budget}} := \{z : \|z\|_\infty \leqslant \rho, \ \|z\|_1 \leqslant \Gamma\rho\}$$

Intuition for Some Basic Uncertainty Sets

- Recall PILOT4; how to build some "safety buffers" for constraint like #372:

$$
\begin{aligned}
&-15.79081 \cdot x_{826} - 8.598819 \cdot x_{827} - 1.88789 \cdot x_{828} - 1.362417 \cdot x_{829} \\
&-1.526049 \cdot x_{830} - 0.031883 \cdot x_{849} - 28.725555 \cdot x_{850} - 10.792065 \cdot x_{851} \\
&-0.19004 \cdot x_{852} - 2.757176 \cdot x_{853} - 12.290832 \cdot x_{854} + 717.562256 \cdot x_{855} \\
&-0.057865x \cdot x_{856} - 3.785417 \cdot x_{857} - 78.30661 \cdot x_{858} - 122.163055 \cdot x_{859} \\
&-6.46609 \cdot x_{860} - 0.48371 \cdot x_{861} - 0.615264 \cdot x_{862} - 1.353783 \cdot x_{863} \\
&-84.644257 \cdot x_{864} - 122.459045 \cdot x_{865} - 43.15593 \cdot x_{866} - 1.712592 \cdot x_{870} \\
&-0.401597 \cdot x_{871} + x_{880} - 0.946049 \cdot x_{898} - 0.946049 \cdot x_{916} \geqslant 23.387405
\end{aligned}
$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$
\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}
$$

- How about a **box** uncertainty set? For some confidence level ρ:

$$
\mathcal{U}_{\text{box}} := \{z : -\rho \leqslant z_i \leqslant \rho\} = \{z : \|z\|_\infty \leqslant \rho\}
$$

- Or maybe an **ellipsoid** would be less conservative:

$$
\mathcal{U}_{\text{ellipsoid}} := \{z : \|z\|_2 \leqslant \rho\}
$$

- Or what if we gave "nature" a **budget** on how many coefficients it could change:

$$
\mathcal{U}_{\text{budget}} := \{z : \|z\|_\infty \leqslant \rho, \ \|z\|_1 \leqslant \Gamma\rho\}
$$

- How to formulate the robust counterpart? How to set ρ, Γ? How to use in practice?

Formulating the Robust Counterpart (RC) for Box Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^\top x \leqslant b, \; \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{box}} = \{z : \|z\|_\infty \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\max_{z : \|z\|_\infty \leqslant \rho} (\bar{a} + Pz)^\top x \leqslant b,$$

Formulating the Robust Counterpart (RC) for Box Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{box}} = \{z : \|z\|_\infty \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\max_{z:\|z\|_\infty \leqslant \rho} (\bar{a} + Pz)^T x \leqslant b,$$

or

$$\bar{a}^T x + \max_{z:\|z\|_\infty \leqslant \rho} (P^T x)^T z \leqslant b,$$

Formulating the Robust Counterpart (RC) for Box Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^\top x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{box}} = \{z : \|z\|_\infty \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\max_{z:\|z\|_\infty \leqslant \rho} (\bar{a} + Pz)^\top x \leqslant b,$$

or

$$\bar{a}^\top x + \max_{z:\|z\|_\infty \leqslant \rho} (P^\top x)^\top z \leqslant b,$$

or

$$\bar{a}^\top x + \max_{z:|z_i| \leqslant \rho} \sum_i (P^\top x)_i \, z_i \leqslant b,$$

Formulating the Robust Counterpart (RC) for Box Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^\top x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{box}} = \{z : \|z\|_\infty \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\max_{z:\|z\|_\infty \leqslant \rho} (\bar{a} + Pz)^\top x \leqslant b,$$

or

$$\bar{a}^\top x + \max_{z:\|z\|_\infty \leqslant \rho} (P^\top x)^\top z \leqslant b,$$

or

$$\bar{a}^\top x + \max_{z:|z_i| \leqslant \rho} \sum_i (P^\top x)_i \, z_i \leqslant b,$$

or

$$\bar{a}^\top x + \rho \sum_i |(P^\top x)_i| \leqslant b,$$

Formulating the Robust Counterpart (RC) for Box Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{\mathbf{a}} + P\mathbf{z})^T \mathbf{x} \leqslant b, \ \forall \, \mathbf{z} \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{box}} = \{\mathbf{z} : \|\mathbf{z}\|_\infty \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\max_{\mathbf{z}:\|\mathbf{z}\|_\infty \leqslant \rho} (\bar{a} + P\mathbf{z})^T x \leqslant b,$$

or

$$\bar{a}^T x + \max_{\mathbf{z}:\|\mathbf{z}\|_\infty \leqslant \rho} (P^T x)^T \mathbf{z} \leqslant b,$$

or

$$\bar{a}^T x + \max_{\mathbf{z}:|z_i| \leqslant \rho} \sum_i (P^T x)_i \, z_i \leqslant b,$$

or

$$\bar{a}^T x + \rho \sum_i |(P^T x)_i| \leqslant b,$$

or

$$\boxed{\bar{a}^T x + \rho \|P^T x\|_1 \leqslant b.}$$

Formulating the Robust Counterpart (RC) for Polyhedral Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^{\top} x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{z : Dz \leqslant d\}$, satisfying the constraint robustly is equivalent to:

Formulating the Robust Counterpart (RC) for Polyhedral Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{\mathbf{a}} + P\boldsymbol{z})^\top \mathbf{x} \leqslant b, \ \forall \boldsymbol{z} \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{\boldsymbol{z} : D\boldsymbol{z} \leqslant \mathbf{d}\}$, satisfying the constraint robustly is equivalent to:

$$\bar{a}^\top x + \max_{z : Dz \leqslant d} (P^\top x)^\top z \leqslant b. \tag{1}$$

Formulating the Robust Counterpart (RC) for Polyhedral Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{z : Dz \leqslant d\}$, satisfying the constraint robustly is equivalent to:

$$\bar{a}^T x + \max_{z : Dz \leqslant d} (P^T x)^T z \leqslant b. \quad (1)$$

By strong LP duality, when the left-hand-side in (1) is finite, we must have:

$$\max\{(P^T x)^T z \ : \ Dz \leqslant d\} = \min\{d^T y : D^T y = P^T x, \ y \geqslant 0\}.$$

Formulating the Robust Counterpart (RC) for Polyhedral Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{z : Dz \leqslant d\}$, satisfying the constraint robustly is equivalent to:

$$\bar{a}^T x + \max_{z : Dz \leqslant d} (P^T x)^T z \leqslant b. \quad (1)$$

By strong LP duality, when the left-hand-side in (1) is finite, we must have:

$$\max\{(P^T x)^T z \ : \ Dz \leqslant d\} = \min\{d^T y : D^T y = P^T x, \ y \geqslant 0\}.$$

Hence (1) is equivalent to

$$\bar{a}^T x + \min_y \{d^T y \ : \ D^T y = P^T x, \ y \geqslant 0\} \leqslant b,$$

Formulating the Robust Counterpart (RC) for Polyhedral Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^\top x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{z : Dz \leqslant d\}$, satisfying the constraint robustly is equivalent to:

$$\bar{a}^\top x + \max_{z : Dz \leqslant d} (P^\top x)^\top z \leqslant b. \tag{1}$$

By strong LP duality, when the left-hand-side in (1) is finite, we must have:

$$\max\{(P^\top x)^\top z \ : \ Dz \leqslant d\} = \min\{d^\top y : D^\top y = P^\top x, \ y \geqslant 0\}.$$

Hence (1) is equivalent to

$$\bar{a}^\top x + \min_y \{d^\top y \ : \ D^\top y = P^\top x, \ y \geqslant 0\} \leqslant b,$$

or

$$\boxed{\exists\, y : \bar{a}^\top x + d^\top y \leqslant b, \quad D^\top y = P^\top x, \quad y \geqslant 0.}$$

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leq b, \ \forall z \in \mathcal{U}} \tag{2}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{z : Dz \leq d\}$, satisfying the constraint robustly is equivalent to:

$$\boxed{\exists y : \bar{a}^T x + d^T y \leq b, \quad D^T y = P^T x, \quad y \geq 0.}$$

Remarks.

- To formulate the RC for (2), we must introduce a set of **auxiliary decision variables** y
  - these are **decision variables**, chosen together with x
- *How many* **auxiliary variables** *are needed to derive the RC for* (2)*?*
- *How many* **constraints** *are needed to derive the RC for* (2)*?*
- *Suppose we were solving* $\min_x\{c^T x : Ax \leq b\}$, *with* $A \in \mathbb{R}^{m \times n}$ *being uncertain. Under* $\mathcal{U}_{\text{polyhedral}}$ *and* $D \in \mathbb{R}^{p \times q}$, *what kind of problem is the RC of this LO, and how large is it?*

Formulating the Robust Counterpart for Polyhedral Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^\top x \leqslant b, \ \forall z \in \mathcal{U}} \tag{2}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{z : Dz \leqslant d\}$, satisfying the constraint robustly is equivalent to:

$$\boxed{\exists y : \bar{a}^\top x + d^\top y \leqslant b, \quad D^\top y = P^\top x, \quad y \geqslant 0.}$$

Remarks.

- To formulate the RC for (2), we must introduce a set of **auxiliary decision variables** y
  - these are **decision variables**, chosen together with x
- *How many* **auxiliary variables** *are needed to derive the RC for* (2)*?*
  - $\#$ rows of D, i.e., as many as the constraints defining $\mathcal{U}_{\text{polyhedral}}$
- *How many* **constraints** *are needed to derive the RC for* (2)*?*
  - $1 + (\#\text{columns of } D) + (\#\text{rows of } D)$
- *Suppose we were solving* $\min_x \{c^\top x : Ax \leqslant b\}$, *with* $A \in \mathbb{R}^{m \times n}$ *being uncertain. Under* $\mathcal{U}_{\text{polyhedral}}$ *and* $D \in \mathbb{R}^{p \times q}$, *what kind of problem is the RC of this LO, and how large is it?*

Formulating the Robust Counterpart for Polyhedral Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}} \tag{2}$$

- For $\mathcal{U}_{\text{polyhedral}} = \{z : Dz \leqslant d\}$, satisfying the constraint robustly is equivalent to:

$$\boxed{\exists y : \bar{a}^T x + d^T y \leqslant b, \quad D^T y = P^T x, \quad y \geqslant 0.}$$

Remarks.

- To formulate the RC for (2), we must introduce a set of **auxiliary decision variables** y
  - these are **decision variables**, chosen together with x
- *How many* **auxiliary variables** *are needed to derive the RC for* (2)*?*
  - $\#$ rows of D, i.e., as many as the constraints defining $\mathcal{U}_{\text{polyhedral}}$
- *How many* **constraints** *are needed to derive the RC for* (2)*?*
  - $1 + (\#\text{columns of } D) + (\#\text{rows of } D)$
- *Suppose we were solving* $\min_x \{c^T x : Ax \leqslant b\}$, *with* $A \in \mathbb{R}^{m \times n}$ *being uncertain. Under* $\mathcal{U}_{\text{polyhedral}}$ *and* $D \in \mathbb{R}^{p \times q}$, *what kind of problem is the RC of this LO, and how large is it?*
  - the RC of a **linear optimization** with $\mathcal{U}_{\text{polyhedral}}$ is still a linear optimization
  - $n + m \cdot p$ variables, $m \cdot (1 + p + q)$ constraints

Formulating the Robust Counterpart (RC) for Ellipsoidal Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{ellipsoid}} = \{z : \|z\|_2 \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

Formulating the Robust Counterpart (RC) for Ellipsoidal Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^\top x \leqslant b, \ \forall\, z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{ellipsoid}} = \{z : \|z\|_2 \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\bar{a}^\top x + \max_{z:\|z\|_2 \leqslant \rho} (P^\top x)^\top z \leqslant b.$$

Formulating the Robust Counterpart (RC) for Ellipsoidal Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^\top x \leqslant b, \ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{ellipsoid}} = \{z : \|z\|_2 \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\bar{a}^\top x + \max_{z:\|z\|_2 \leqslant \rho} (P^\top x)^\top z \leqslant b.$$

Intermezzo: $\max \{q^\top z : \|z\|_2 \leqslant \rho\}$ *or* $\max \{q^\top z : z^\top z \leqslant \rho^2\}$

Lagrange: $z = q/\lambda$, *and* $\lambda = \|q\|_2/\rho$.

Optimal objective value: $\frac{q^\top q}{\lambda} = \rho\|q\|_2$.

Formulating the Robust Counterpart (RC) for Ellipsoidal Uncertainty Set

- Consider a **linear constraint** in x with coefficients that depend **linearly** on z

$$\boxed{(\bar{a} + Pz)^T x \leqslant b,\ \forall z \in \mathcal{U}}$$

- For $\mathcal{U}_{\text{ellipsoid}} = \{z : \|z\|_2 \leqslant \rho\}$, satisfying the constraint robustly is equivalent to:

$$\bar{a}^T x + \max_{z:\|z\|_2 \leqslant \rho} (P^T x)^T z \leqslant b.$$

Intermezzo: max $\{q^T z : \|z\|_2 \leqslant \rho\}$ *or* max $\{q^T z : z^T z \leqslant \rho^2\}$

Lagrange: $z = q/\lambda$, *and* $\lambda = \|q\|_2/\rho$.

Optimal objective value: $\frac{q^T q}{\lambda} = \rho\|q\|_2$.

Hence robust counterpart (RC) is:

$$\boxed{\bar{a}^T x + \rho\|P^T x\|_2 \leqslant b.}$$

RC for **Linear Optimization** Problems with Classical Sets

The robust counterpart for $(\bar{a} + Pz)^T x \leqslant b, \; \forall z \in \mathcal{U}$ is:

U-set	$\mathcal{U}$	**Robust Counterpart**	**Tractability**
Box	$\|z\|_\infty \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_1 \leqslant b$	LO
Ellipsoidal	$\|z\|_2 \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_2 \leqslant b$	CQO
Polyhedral	$Dz \leqslant d$	$\exists y : \begin{cases} \bar{a}^T x + d^T y \leqslant b \\ D^T y = P^T x \\ y \geqslant 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leqslant \rho \\ \|z\|_1 \leqslant \Gamma \end{cases}$	$\exists y : \bar{a}^T x + \rho \|y\|_1 + \Gamma \|P^T x - y\|_\infty \leqslant b$	LO

RC for **Linear Optimization** Problems with Classical Sets

The robust counterpart for $(\bar{a} + Pz)^T x \leqslant b, \ \forall z \in \mathcal{U}$ is:

U-set	$\mathcal{U}$	Robust Counterpart	Tractability
Box	$\|z\|_\infty \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_1 \leqslant b$	LO
Ellipsoidal	$\|z\|_2 \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_2 \leqslant b$	CQO
Polyhedral	$Dz \leqslant d$	$\exists y : \begin{cases} \bar{a}^T x + d^T y \leqslant b \\ D^T y = P^T x \\ y \geqslant 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leqslant \rho \\ \|z\|_1 \leqslant \Gamma \end{cases}$	$\exists y : \bar{a}^T x + \rho \|y\|_1 + \Gamma \|P^T x - y\|_\infty \leqslant b$	LO

- Problems above can be handled by large-scale modern solvers: CPLEX, Gurobi, etc.
- Some software now also handling automatic problem re-formulation
- If some of the decisions x are integer, problems above become MI-LO/CQO
- Already a lot of mileage in many practical problems:
  logistics and supply chain management, radiation therapy, scheduling, ...

RC for **Linear Optimization** Problems with Classical Sets

The robust counterpart for $(\bar{a} + Pz)^\top x \leqslant b, \ \forall z \in \mathcal{U}$ is:

U-set	$\mathcal{U}$	**Robust Counterpart**	**Tractability**
Box	$\|z\|_\infty \leqslant \rho$	$\bar{a}^\top x + \rho \|P^\top x\|_1 \leqslant b$	LO
Ellipsoidal	$\|z\|_2 \leqslant \rho$	$\bar{a}^\top x + \rho \|P^\top x\|_2 \leqslant b$	CQO
Polyhedral	$Dz \leqslant d$	$\exists y : \begin{cases} \bar{a}^\top x + d^\top y \leqslant b \\ D^\top y = P^\top x \\ y \geqslant 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leqslant \rho \\ \|z\|_1 \leqslant \Gamma \end{cases}$	$\exists y : \bar{a}^\top x + \rho \|y\|_1 + \Gamma \|P^\top x - y\|_\infty \leqslant b$	LO

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^\top x \leqslant b + p^\top z, \ \forall z \in \mathcal{U}$
  $\Leftrightarrow \bar{a}^\top x + (P^\top x - p)^\top z \leqslant b, \ \forall z \in \mathcal{U}$, so can use base model

RC for **Linear Optimization** Problems with Classical Sets

The robust counterpart for $(\bar{a} + Pz)^T x \leqslant b,\ \forall z \in \mathcal{U}$ is:

U-set	$\mathcal{U}$	Robust Counterpart	Tractability
Box	$\|z\|_\infty \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_1 \leqslant b$	LO
Ellipsoidal	$\|z\|_2 \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_2 \leqslant b$	CQO
Polyhedral	$Dz \leqslant d$	$\exists y : \begin{cases} \bar{a}^T x + d^T y \leqslant b \\ D^T y = P^T x \\ y \geqslant 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leqslant \rho \\ \|z\|_1 \leqslant \Gamma \end{cases}$	$\exists y : \bar{a}^T x + \rho \|y\|_1 + \Gamma \|P^T x - y\|_\infty \leqslant b$	LO

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^T x \leqslant b + p^T z,\ \forall z \in \mathcal{U}$
  $\Leftrightarrow \bar{a}^T x + (P^T x - p)^T z \leqslant b,\ \forall z \in \mathcal{U}$, so can use base model
- **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leqslant 0,\ k \in K\}$, $h_k(\cdot)$ convex?
  RC is $\exists \{w_k, u_k\}_{k \in K} : \bar{a}^T x + \sum_k u_k h_k^\star(w_k/u_k) \leqslant b,\ \sum_k w^k = P^T x,\ u \geqslant 0.$ $h_k^\star$ is convex conjugate of h_k

RC for **Linear Optimization** Problems with Classical Sets

The robust counterpart for $(\bar{a} + Pz)^T x \leq b, \; \forall z \in \mathcal{U}$ is:

U-set	$\mathcal{U}$	Robust Counterpart	Tractability
Box	$\lVert z \rVert_\infty \leq \rho$	$\bar{a}^T x + \rho \lVert P^T x \rVert_1 \leq b$	LO
Ellipsoidal	$\lVert z \rVert_2 \leq \rho$	$\bar{a}^T x + \rho \lVert P^T x \rVert_2 \leq b$	CQO
Polyhedral	$Dz \leq d$	$\exists y : \begin{cases} \bar{a}^T x + d^T y \leq b \\ D^T y = P^T x \\ y \geq 0 \end{cases}$	LO
Budget	$\begin{cases} \lVert z \rVert_\infty \leq \rho \\ \lVert z \rVert_1 \leq \Gamma \end{cases}$	$\exists y : \bar{a}^T x + \rho \lVert y \rVert_1 + \Gamma \lVert P^T x - y \rVert_\infty \leq b$	LO

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^T x \leq b + p^T z, \; \forall z \in \mathcal{U}$
  $\Leftrightarrow \bar{a}^T x + (P^T x - p)^T z \leq b, \; \forall z \in \mathcal{U}$, so can use base model
- **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leq 0, \; k \in K\}$, $h_k(\cdot)$ convex?
  RC is $\exists \{w_k, u_k\}_{k \in K} : \bar{a}^T x + \sum_k u_k h_k^\star(w_k / u_k) \leq b, \; \sum_k w^k = P^T x, \; u \geq 0$. $h_k^\star$ is convex conjugate of h_k
- **Constraint LHS general in x, linear in z:** $(Pz)^T g(x) \leq b, \; \forall z \in \mathcal{U}$
  To calculate RC, take $\bar{a} = 0$ and replace x with $g(x)$ in our base-case model

RC for **Linear Optimization** Problems with Classical Sets

The robust counterpart for $(\bar{a} + Pz)^T x \leqslant b,\ \forall z \in \mathcal{U}$ is:

U-set	$\mathcal{U}$	Robust Counterpart	Tractability
Box	$\|z\|_\infty \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_1 \leqslant b$	LO
Ellipsoidal	$\|z\|_2 \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_2 \leqslant b$	CQO
Polyhedral	$Dz \leqslant d$	$\exists y : \begin{cases} \bar{a}^T x + d^T y \leqslant b \\ D^T y = P^T x \\ y \geqslant 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leqslant \rho \\ \|z\|_1 \leqslant \Gamma \end{cases}$	$\exists y : \bar{a}^T x + \rho \|y\|_1 + \Gamma \|P^T x - y\|_\infty \leqslant b$	LO

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^T x \leqslant b + p^T z,\ \forall z \in \mathcal{U}$
  $\Leftrightarrow \bar{a}^T x + (P^T x - p)^T z \leqslant b,\ \forall z \in \mathcal{U}$, so can use base model
- **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leqslant 0,\ k \in K\}$, $h_k(\cdot)$ convex?
  RC is $\exists \{w_k, u_k\}_{k \in K} : \bar{a}^T x + \sum_k u_k h_k^\star(w_k/u_k) \leqslant b,\ \sum_k w^k = P^T x,\ u \geqslant 0$. $h_k^\star$ is convex conjugate of h_k
- **Constraint LHS general in x, linear in z:** $(Pz)^T g(x) \leqslant b,\ \forall z \in \mathcal{U}$
  To calculate RC, take $\bar{a} = 0$ and replace x with $g(x)$ in our base-case model
- **Constraint LHS linear in $x \geqslant 0$, concave in z:** $x^T g(\bar{a} + Pz) \leqslant b,\ \forall z \in \mathcal{U},\ g_i(y)$ concave
  $\Leftrightarrow d^T x \leqslant b,\ \forall (z, d) \in \mathcal{U}^+ := \{(z, d) \mid \exists a : a = \bar{a} + Pz,\ d \leqslant f(a),\ z \in \mathcal{U}\}$; now **linear** in (z, d), and $\mathcal{U}^+$ convex

RC for **Linear Optimization** Problems with Classical Sets

The robust counterpart for $\boxed{(\bar{a} + Pz)^T x \leqslant b,\ \forall z \in \mathcal{U}}$ is:

U-set	$\mathcal{U}$	Robust Counterpart	Tractability
Box	$\|z\|_\infty \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_1 \leqslant b$	LO
Ellipsoidal	$\|z\|_2 \leqslant \rho$	$\bar{a}^T x + \rho \|P^T x\|_2 \leqslant b$	CQO
Polyhedral	$Dz \leqslant d$	$\exists y : \begin{cases} \bar{a}^T x + d^T y \leqslant b \\ D^T y = P^T x \\ y \geqslant 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leqslant \rho \\ \|z\|_1 \leqslant \Gamma \end{cases}$	$\exists y : \bar{a}^T x + \rho \|y\|_1 + \Gamma \|P^T x - y\|_\infty \leqslant b$	LO

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^T x \leqslant b + p^T z,\ \forall z \in \mathcal{U}$
  $\Leftrightarrow \bar{a}^T x + (P^T x - p)^T z \leqslant b,\ \forall z \in \mathcal{U}$, so can use base model
- **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leqslant 0,\ k \in K\}$, $h_k(\cdot)$ convex?
  RC is $\exists \{w_k, u_k\}_{k \in K} : \bar{a}^T x + \sum_k u_k h_k^\star(w_k / u_k) \leqslant b,\ \sum_k w^k = P^T x,\ u \geqslant 0$. $h_k^\star$ is convex conjugate of h_k
- **Constraint LHS general in x, linear in z:** $(Pz)^T g(x) \leqslant b,\ \forall z \in \mathcal{U}$
  To calculate RC, take $\bar{a} = 0$ and replace x with $g(x)$ in our base-case model
- **Constraint LHS linear in $x \geqslant 0$, concave in z:** $x^T g(\bar{a} + Pz) \leqslant b,\ \forall z \in \mathcal{U}$, $g_i(y)$ concave
  $\Leftrightarrow d^T x \leqslant b,\ \forall (z, d) \in \mathcal{U}^+ := \{(z, d) \mid \exists a : a = \bar{a} + Pz,\ d \leqslant f(a),\ z \in \mathcal{U}\}$; now **linear** in (z, d), and $\mathcal{U}^+$ convex
- **Constraint LHS convex in x and convex in z:** $f(x, z) \leqslant b$, f jointly convex
  Tractable if f has "easy" piece-wise description: $f(x, z) = \max_{k \in K} f_k(x, z)$, where f_k are cases that "worked"

Extensions

- **Uncertainty in the right-hand side:** $(\bar{\mathbf{a}} + P\boldsymbol{z})^\top \boldsymbol{x} \leqslant b + \mathbf{p}^\top \boldsymbol{z},\ \forall\, \boldsymbol{z} \in \mathcal{U}$

  $\Leftrightarrow \bar{\mathbf{a}}^\top \boldsymbol{x} + (P^\top \boldsymbol{x} - p)^\top \boldsymbol{z} \leqslant b,\ \forall\, \boldsymbol{z} \in \mathcal{U}$, so can use base model

Extensions

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^\top x \leqslant b + p^\top z,\ \forall z \in \mathcal{U}$

  $\Leftrightarrow \bar{a}^\top x + (P^\top x - p)^\top z \leqslant b,\ \forall z \in \mathcal{U}$, so can use base model

- **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leqslant 0,\ k \in K\}$, $h_k(\cdot)$ convex?

  $\Leftrightarrow \quad \exists \{w_k, u_k\}_{k\in K} : \bar{a}^\top x + \sum_k u_k h_k^\star(w_k/u_k) \leqslant b,\ \sum_k w^k = P^\top x,\ u \geqslant 0.$

  $h_k^\star$ is convex conjugate of h_k

Extensions

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^\top x \leqslant b + p^\top z,\ \forall z \in \mathcal{U}$

  $\Leftrightarrow \bar{a}^\top x + (P^\top x - p)^\top z \leqslant b,\ \forall z \in \mathcal{U}$, so can use base model

- **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leqslant 0,\ k \in K\}$, $h_k(\cdot)$ convex?

  $\Leftrightarrow \quad \exists \{w_k, u_k\}_{k \in K} : \bar{a}^\top x + \sum_k u_k h_k^\star(w_k/u_k) \leqslant b,\ \sum_k w^k = P^\top x,\ u \geqslant 0.$

  $h_k^\star$ is convex conjugate of h_k

- **LHS general in x, linear in z:** $(Pz)^\top g(x) \leqslant b,\ \forall z \in \mathcal{U}$

  To calculate RC, take $\bar{a} = 0$ and replace x with $g(x)$ in our base-case model

Extensions

- **Uncertainty in the right-hand side:** $(\bar{a} + P\mathbf{z})^T \mathbf{x} \leqslant b + \mathbf{p}^T \mathbf{z}, \ \forall \mathbf{z} \in \mathcal{U}$

  $\Leftrightarrow \bar{\mathbf{a}}^T \mathbf{x} + (P^T \mathbf{x} - p)^T \mathbf{z} \leqslant b, \ \forall \mathbf{z} \in \mathcal{U}$, so can use base model

- **General convex uncertainty set:** $\mathcal{U} = \{\mathbf{z} : h_k(\mathbf{z}) \leqslant 0, \ k \in K\}$, $h_k(\cdot)$ convex?

  $\Leftrightarrow \quad \exists \{\mathbf{w}_k, u_k\}_{k \in K} : \bar{\mathbf{a}}^T \mathbf{x} + \sum_k \mathbf{u}_k h_k^\star(\mathbf{w}_k / u_k) \leqslant b, \ \sum_k \mathbf{w}^k = P^T x, \ \mathbf{u} \geqslant 0.$

  $h_k^\star$ is convex conjugate of h_k

- **LHS general in x, linear in z:** $(P\mathbf{z})^T g(\mathbf{x}) \leqslant b, \ \forall \mathbf{z} \in \mathcal{U}$

  To calculate RC, take $\bar{\mathbf{a}} = 0$ and replace $\mathbf{x}$ with $g(\mathbf{x})$ in our base-case model

- **LHS linear in $x \geqslant 0$, concave in z:** $x^T g(\bar{a} + Pz) \leqslant b, \ \forall z \in \mathcal{U}$, g concave

  $\Leftrightarrow d^T x \leqslant b, \ \forall (z, d) \in \mathcal{U}^+ := \left\{(z, d) \mid \exists a : a = \bar{a} + Pz, \ d \leqslant f(a), \ z \in \mathcal{U}\right\}$

  now **linear** in (z, d), and $\mathcal{U}^+$ convex

- **Uncertainty in the right-hand side:** $(\bar{a} + Pz)^T x \leqslant b + p^T z, \ \forall z \in \mathcal{U}$

  $\Leftrightarrow \bar{a}^T x + (P^T x - p)^T z \leqslant b, \ \forall z \in \mathcal{U}$, so can use base model

- **General convex uncertainty set:** $\mathcal{U} = \{z : h_k(z) \leqslant 0, \ k \in K\}$, $h_k(\cdot)$ convex?

  $\Leftrightarrow \quad \exists \{w_k, u_k\}_{k \in K} : \bar{a}^T x + \sum_k u_k h_k^\star(w_k / u_k) \leqslant b, \ \sum_k w^k = P^T x, \ u \geqslant 0.$

  $h_k^\star$ is convex conjugate of h_k

- **LHS general in x, linear in z:** $(Pz)^T g(x) \leqslant b, \ \forall z \in \mathcal{U}$

  To calculate RC, take $\bar{a} = 0$ and replace x with $g(x)$ in our base-case model

- **LHS linear in $x \geqslant 0$, concave in z:** $x^T g(\bar{a} + Pz) \leqslant b, \ \forall z \in \mathcal{U}$, g concave

  $\Leftrightarrow d^T x \leqslant b, \ \forall (z, d) \in \mathcal{U}^+ := \left\{(z, d) \mid \exists a : a = \bar{a} + Pz, \ d \leqslant f(a), \ z \in \mathcal{U}\right\}$

  now **linear** in (z, d), and $\mathcal{U}^+$ convex

- **LHS convex in x and convex in z:** $f(x, z) \leqslant b$, f jointly convex

  Tractable if f has "easy" piece-wise description: $f(x, z) = \max_{k \in K} f_k(x, z)$, where f_k are cases that "worked"

Used in many applications

- inventory management e.g., [Ben-Tal et al., 2005, Bertsimas and Thiele, 2006, Bienstock and Özbay, 2008, ...]
- facility location and transportation [Baron et al., 2011, ...]
- scheduling [Lin et al., 2004, Yamashita et al., 2007, Mittal et al., 2014, ...]
- revenue management [Perakis and Roels, 2010, Adida and Perakis, 2006, ...]
- project management [Wiesemann et al., 2012, Ben-Tal et al., 2009, ...]
- energy generation and distribution [Zhao et al., 2013, Lorca and Sun, 2015, ...]
- portfolio optimization [Goldfarb and Iyengar, 2003, Tütüncü and Koenig, 2004, Ceria and Stubbs, 2006, Pinar and Tütüncü, 2005, Bertsimas and Pachamanova, 2008, ...]
- healthcare [Borfeld et al., 2008, Hanne et al., 2009, Chen et al., 2011, I., Trichakis, Yoon (2018), ...]
- humanitarian [Uichano 2017, den Hertog et al., 2019, ...]

E. Adida and G. Perakis. A robust optimization approach to dynamic pricing and inventory control with no backorders. *Mathematical Programming*, 107: 97–129, 2006.

O. Baron, J. Milner, and H. Naseraldin. Facility location: A robust optimization approach. *Production and Operations Management*, 20(5):772–785, 2011.

A. Ben-Tal, A. Goryashko, E. Guslitzer, and A. Nemirovski. Adjustable robust solutions of uncertain linear programs. *Mathematical Programming*, 99(2): 351–376, 2004.

A. Ben-Tal, B. Golany, A. Nemirovski, and J.-P. Vial. Retailer-supplier flexible commitments contracts: A robust optimization approach. *Manufacturing & Service Operations Management*, 7(3):248–271, 2005.

A. Ben-Tal, L. El-Ghaoui, and A. Nemirovski. *Robust Optimization*. Princeton Series in Applied Mathematics. Princeton University Press, 2009.

D. Bertsimas and D. Pachamanova. Robust multiperiod portfolio management in the presence of transaction costs. *Computers and Operations Research*, 35(1):3–17, 2008.

D. Bertsimas and A. Thiele. A robust optimization approach to inventory theory. *Operations Research*, 54(1):150–168, 2006.

D. Bienstock and N. Özbay. Computing robust basestock levels. *Discrete Optimization*, 5(2):389 – 414, 2008.

S. Ceria and R. Stubbs. Incorporating estimation errors into portfolio selection: Robust portfolio construction. *Journal of Asset Management*, 7(2): 109–127, July 2006.

F. de Ruiter, A. Ben-Tal, R. Brekelmans, and D. den Hertog. Adjustable robust optimization with decision rules based on inexact revealed data. Center discussion paper series no. 2014-003, CentER, 2014.

D. Goldfarb and G. Iyengar. Robust Portfolio Selection Problems. *Mathematics of Operations Research*, 28(1):1–38, 2003.

X. Lin, S. Janak, and C. Floudas. A new robust optimization approach for scheduling under uncertainty: I. bounded uncertainty. *Computers and Chemical Engineering*, 28:1069–1085, 2004.

A. Lorca and A. Sun. Adaptive robust optimization with dynamic uncertainty sets for multi-period economic dispatch under significant wind. *IEEE Transactions on Power Systems*, 30(4):1702–1713, 2015.

S. Mittal, A. Schulz, and S. Stiller. Robust appointment scheduling. In *APPROX*, 2014.

G. Perakis and G. Roels. Robust controls for network revenue management. *Manufacturing & Service Operations Management*, 12(1):56–76, 2010.

M. Pinar and R. Tütüncü. Robust profit opportunities in risky financial portfolios. *Operations Research Letters*, 33(4):331 – 340, 2005.

A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming*. MPS / SIAM Series on Optimization. SIAM, 2009.

R. Tütüncü and M. Koenig. Robust asset allocation. *Annals of Operations Research*, 132(1-4):157–187, November 2004.

W. Wiesemann, D. Kuhn, and B. Rustem. Robust resource allocations in temporal networks. *Mathematical Programming*, 135:437–471, 2012. ISSN 0025-5610.

D. Yamashita, V. Armentano, and M. Laguna. Robust optimization models for project scheduling with resource availability cost. *Journal of Scheduling*, 10 (1):67–76, 2007.

C. Zhao, J.Wang, J. Watson, and Y.Guan. Multi-stage robust unit commitment considering wind and demand response uncertainties. *IEEE Transactions on Power Systems*, 28(3):2708–2717, 2013.