Introduction to Robust Optimization

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Outline for Today

1. Robust Optimization
   - Quick Recap
   - Calibrating Uncertainty Sets
   - Distributionally Robust Optimization
   - Connections with Other Areas
   - Some Important Caveats When Applying Robust Optimization

2. Dynamic Robust Optimization
   - Properly Writing a Robust DP
   - An Inventory Example
   - Tractable Approximations with Decision Rules
   - Some Practical Issues
   - Bellman Optimality
   - An Application in Monitoring

"Classical" Robust Optimization (RO)

- Only information about unknowns z: they belong to an **uncertainty set** $\mathcal{U}$
- Solve the following optimization problem:

$$(P) \qquad \begin{aligned} & \inf_{x} \sup_{z \in \mathcal{U}} C(x, z) \\ & \text{s.t. } f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}, \forall i \in I \end{aligned}$$

- This model has **infinitely many** constraints
- W.l.o.g., we can consider uncertainty only in the constraints
- Each and every constraint must satisfied: $f_i(x, z) \leqslant 0, \forall z \in \mathcal{U}$
- How to reformulate this as a **finite-dimensional, tractable** optimization problem, a.k.a. the **robust counterpart**?

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U-set	$\mathcal{U}$	Robust Counterpart	Tractability
Box	$\|z\|_\infty \leqslant \rho$	$\bar{a}^\intercal x + \rho \|P^\intercal x\|_1 \leqslant b$	LO
Ellipsoidal	$\|z\|_2 \leqslant \rho$	$\bar{a}^\intercal x + \rho \|P^\intercal x\|_2 \leqslant b$	CQO
Polyhedral	$Dz \leqslant d$	$\exists y : \begin{cases} \bar{a}^\intercal x + d^\intercal y \leqslant b \\ D^\intercal y = P^\intercal x \\ y \geqslant 0 \end{cases}$	LO
Budget	$\begin{cases} \|z\|_\infty \leqslant \rho \\ \|z\|_1 \leqslant \Gamma \end{cases}$	$\exists y : \bar{a}^\intercal x + \rho \|y\|_1 + \Gamma \|P^\intercal x - y\|_\infty \leqslant b$	LO
Convex	$h_k(z) \leqslant 0$	$\begin{cases} a^\intercal x + \sum_k u_k h_k^* \left(\frac{w^k}{u_k}\right) \leqslant b \\ \sum_k w^k = P^\intercal x \\ u \geqslant 0 \end{cases}$	Conv. Opt.

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- Several extensions
- Robust counterparts can be handled by large-scale modern solvers
- Already a lot of mileage in many practical problems:
  logistics and supply chain management, radiation therapy, scheduling, ...

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- How to pick parameters like ρ, Γ?
- How to build uncertainty sets (from data)?

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Suppose z_i *are really random, and we seek* ρ, Γ *to ensure* $\mathbb{P}\big[(\bar{a} + P\boldsymbol{z})^\intercal x \leqslant b\big]$ *is "large",* $\forall\, \mathbb{P}$

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Suppose z_i *are* **independent r.v. with mean 0 and support on** $[-1, 1]$. *Then:*

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- *if x feasible for RC with* $\mathcal{U}_{\text{budget}} = \{z \in \mathbb{R}^L : \|z\|_\infty \leqslant 1,\ \|z\|_1 \leqslant \Gamma = \sqrt{2\ln(1/\epsilon)}\sqrt{L}\}$,
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- Some probabilistic information allows controlling conservatism: **very useful in applications**
- The budget Γ depends on the dimension of z (L), whereas ρ does not!
- Proofs based on concentration inequalities

Another Quick Example: A Portfolio Problem (Ben-Tal and Nemirovski)

- 200 risky assets; asset # 200 is cash, with yearly return $r_{200} = 5\%$ and zero risk
- Yearly returns r_i are **independent r.v.** with values in $[\mu_i - \sigma_i, \mu_i + \sigma_i]$ and means μ_i:

$$\mu_i = 1.05 + 0.3\frac{(200-i)}{199}, \quad \sigma_i = 0.05 + 0.6\frac{(200-i)}{199}, \quad i = 1, \ldots, 199.$$

- Goal: distribute \$1 so as to maximize worst-case value-at-risk at level $\epsilon = 0.5\%$:

$$\max_{x,t}\left\{t \ : \ \mathbb{P}\left[\sum_{i=1}^{199} r_i x_i + r_{200}x_{200} \geqslant t\right] \geqslant 1-\epsilon, \ \forall\, \mathbb{P}, \ \sum_{i=1}^{200} x_i = 1, \ x \geqslant 0\right\},$$

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- With $z_i \stackrel{\text{def}}{=} (r_i - \mu_i)/\sigma_i$, let's consider 3 uncertainty sets:
  1. $\mathcal{U}_{\text{box}} = \{z : \|z\|_\infty \leqslant 1\}$
  2. $\mathcal{U}_{\text{ellipsoid-box}} = \{z : \|z\|_\infty \leqslant 1, \|z\|_2 \leqslant \rho\}$, with $\rho = \sqrt{2\ln(1/\epsilon)} = 3.255$
  3. $\mathcal{U}_{\text{budget}} = \{z : \|z\|_\infty \leqslant 1, \|z\|_1 \leqslant \Gamma\}$ with $\Gamma = \sqrt{2\ln(1/\epsilon)}\sqrt{199} = 45.921$.

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- Results:
  - $\mathcal{U}_{\text{box}}$: worst-case returns $r_i = \mu_i - \sigma_i$ yield less than risk-free return of 5%, so optimal to keep all money in cash; robust optimal return 1.05, risk 0
  - $\mathcal{U}_{\text{ellipsoid-box}}$: robust optimal value is 1.12, risk 0.5%
  - $\mathcal{U}_{\text{budget}}$: robust optimal value is 1.10, risk 0.5%
- $\mathcal{U}_{\text{box}}$ can be quite conservative, a tiny bit of risk can go a long way...

Using Concentration Results to Model Uncertainty Sets

- Bertsimas & Bandi: let's use the **implications** of the **Central Limit Theorem**

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- Suppose we have uncertainties $\{X_i\}_{i=1}^n$, each with mean μ, standard deviation σ

$$\mathcal{U}_{\text{CLT}} \stackrel{\text{def}}{=} \left\{(x_1, \ldots, x_n) : \left|\sum_{i=1}^n x_i - n\mu\right| \leqslant \Gamma\sigma\sqrt{n}\right\}.$$

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- Many extensions possible
  - Modeling correlations through a factor model:
    $\mathcal{U}_{\text{corr}} \stackrel{\text{def}}{=} \left\{ \boldsymbol{x} : \boldsymbol{x} = P\boldsymbol{z} + \boldsymbol{\epsilon}, \left| \sum_{i=1}^{m} z_i - m\mu_y \right| \leqslant \Gamma \sigma_z \sqrt{m}, \left| \sum_{i=1}^{n} \epsilon_i \right| \leqslant \Gamma \sigma_\epsilon \sqrt{n}, \right\}.$
  - Using stable laws to model heavy-tailed cases where variance is undefined:
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  - Constructing typical sets: if H_f is the (Shannon) entropy of f,
    (i) $\mathbb{P}[\tilde{z} \in \mathcal{U}_{\text{typical}}] \to 1$, (ii) $\left| \frac{1}{n} \log f\left(\tilde{z} | \tilde{z} \in \mathcal{U}_{\text{typical}}\right) + H_f \right| \leqslant \varepsilon_n$

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- Bertsimas & Bandi used these to derive **robust equivalents** for several classical queueing theory and information theory results

Using Hypothesis Tests to Model Uncertainty Sets

Another powerful idea: derive **data-driven** uncertainty sets from **hypothesis tests**

From Bertsimas, Gupta, Kallus (2017):

Table 1 Summary of data-driven uncertainty sets proposed in this paper. SOC, EC and LMI denote second-order cone representable sets, exponential cone representable sets, and linear matrix inequalities, respectively

Assumptions on $\mathbb{P}^*$	Hypothesis test	Geometric description	Eqs.	Inner problem
Discrete support	χ^2-test	SOC	(13, 15)	
Discrete support	G-test	Polyhedral*	(13, 16)	
Independent marginals	KS Test	Polyhedral*	(21)	Line search
Independent marginals	K Test	Polyhedral*	(76)	Line search
Independent marginals	CvM Test	SOC*	(76, 69)	
Independent marginals	W Test	SOC*	(76, 70)	
Independent marginals	AD Test	EC	(76, 71)	
Independent marginals	Chen et al. [23]	SOC	(27)	Closed-form
None	Marginal Samples	Box	(31)	Closed-form
None	Linear Convex Ordering	Polyhedron	(34)	
None	Shawe-Taylor and Cristianini [46]	SOC	(39)	Closed-form
None	Delage and Ye [25]	LMI	(41)	

The additional "*" notation indicates a set of the above type with one additional, relative entropy constraint. KS, K, CvM, W, and AD denote the Kolmogorov–Smirnov, Kuiper, Cramer-von Mises, Watson and Anderson-Darling goodness of fit tests, respectively. In some cases, we can identify a worst-case realization of $\mathbf{u}$ in (1) for bi-affine f and a candidate $\mathbf{x}$ with a specialized algorithm. In these cases, the column "Inner Problem" roughly describes this algorithm

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$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[f(x, \tilde{z})] \leqslant b$$

- Now, the adversary is choosing $\mathbb{P}$, instead of z
  - **Advantage:** $\mathbb{E}_{\mathbb{P}}[f(x, \tilde{z})]$ as an expression of $\mathbb{P}$ is **always linear**, so much of our earlier machinery (e.g., convex duality) can be applied if the set $\mathcal{P}$ is "well-behaved"
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  - **Disadvantage:** Maximizing over continuous $\mathbb{P}$: -dimensional optimization
- Very old idea, dating to the 1950s (Scarf 1958, Zackova 1966)
- Kuhn, Shafiee, Wiesemann (2024): tutorial on state-of-the-art. Can model:
  - known (**bounds** on) moments, e.g., means, covariance matrix, higher order
  - known (**bounds on**) quantiles (e.g., median) or spread statistics
  - multiple confidence regions
  - distance from a nominal distribution (Kullback-Leibler, Wasserstein, etc.)

Two Important Caveats When Working With Robust Models

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\max_{X,I,Z,P} \quad & \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}}\sum_{j\in\mathcal{N}}(p-c^s_{ij})X_{ij\tau} - \sum_{\tau\in\mathcal{T}}\sum_{i\in\mathcal{F}} c_i P_{i\tau} - \sum_{i\in\mathcal{F}}(C_i Z_i + K_i I_i) \\
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$$

Parameters:
$\mathcal{T}$: discrete planning horizon, indexed by τ
$\mathcal{F}$: potential facility locations, indexed by i
$\mathcal{N}$: demand node locations, indexed by j
p: unit price of goods
c_i: cost per unit of production at facility i
C_i: cost per unit of capacity for facility i
K_i: cost of opening a facility at location i
c^s_{ij}: cost of shipping units from location i to j
$D_{j\tau}$: demand in period τ at location j.

Decision variables:
$X_{ij\tau}$: how much of demand j in period τ satisfied by i
$P_{i\tau}$: quantity produced at facility i in period τ
I_i: whether facility i is open (0/1)
Z_i: capacity of facility i if open.

An Example: A Facility Location Problem (Baron et al. 2011)

Need to decide where to open facilities, how much capacity to install, and how to assign customer demands over a future planning horizon, in order to maximize profits.

Step 1. Start with a **deterministic** model formulation:

$$\begin{aligned}
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\end{aligned}$$

Step 2. Identify all uncertain parameters and model the uncertainty set $\mathcal{U}$.

Baron et al. 2011 captured uncertain demands:

$$\mathcal{U} = \left\{ D \in \mathbb{R}^{|\mathcal{N}| \cdot |\mathcal{T}|} \;\middle|\; \sum_{j \in \mathcal{N}} \sum_{t \in \mathcal{T}} \left(\frac{D_{jt} - \bar{D}_{jt}}{\epsilon_t \bar{D}_{jt}} \right)^2 \leqslant \rho^2 \right\},$$

$\{\bar{D}_{jt}\}_{j \in \mathcal{N}; t \in \mathcal{T}}$ are "nominal" demands, ϵ_t is allowed deviation (%), ρ is the size of the ellipsoid.

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Equivalently, can write $D_{jt} = \bar{D}_{jt}(1+\varepsilon_t\cdot z_{jt})$, where $z \in \mathcal{U} = \{z\in\mathbb{R}^{|\mathcal{N}|\cdot|\mathcal{T}|} : \|z\|_2 \leqslant \rho\}$

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Step 3. Derive robust counterpart for the problem. Here, this will be a Conic Quadratic program.

Compare Two Models

Our initial model, with **decisions for quantities** X:

$$
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For fixed D, are these **deterministic/nominal** models **equivalent**?
Are their **robust counterparts** **equivalent**?

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An equivalent **deterministic** model, with **decisions for fractions of demands** Y:

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\end{aligned}$$

The second formulation implements ordering quantities that **depend on demand**!

The **robust counterparts** of **equivalent** deterministic models
may be different!

You should always try to allow your formulation
to be as flexible as possible!

Another Caveat...

Are Robust Solutions **Pareto-Efficient**?

$$\max_{x \in \mathcal{X}} \quad \min_{u \in \mathcal{U}} u^\intercal x$$

- Feasible set of solutions $\mathcal{X} = \left\{x \in \mathbb{R}^n : Ax \leqslant b\right\}$
- Uncertainty set of objective coefficients $\mathcal{U} = \{u \in \mathbb{R}^n : Du \geqslant d\}$

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- Classical RO framework results in
  - Optimal value $J^\star_{\mathsf{RO}}$
  - Set of robustly optimal solutions

$$X^{\mathsf{RO}} = \{x \in \mathcal{X} : \exists\, y \geqslant 0 \text{ such that } D^\intercal y = x, \quad y^\intercal d \geqslant J^\star_{\mathsf{RO}}\}$$

Set of **Robustly Optimal** Solutions

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- What if an uncertainty scenario materializes that does not correspond to the worst-case?

- Are there any guarantees that no other solution $\bar{x}$ exists that, apart from protecting us from worst-case scenarios, also performs better overall, under all possible uncertainty realizations?

Pareto Robustly Optimal solutions (Iancu & Trichakis 2014)

$$\max_{x \in \mathcal{X}} \quad \min_{u \in \mathcal{U}} u^\intercal x \tag{1}$$

Definition

A solution x is called a **Pareto Robustly Optimal (PRO) solution** for Problem (1) if

(a) it is robustly optimal, i.e., $x \in X^{\mathrm{RO}}$, and

(b) there is no $\bar{x} \in \mathcal{X}$ such that

$$\begin{aligned} u^\intercal \bar{x} &\geqslant u^\intercal x, \quad \forall u \in \mathcal{U}, \quad \text{and} \\ \bar{u}^\intercal \bar{x} &> \bar{u}^\intercal x, \ \text{for some } \bar{u} \in \mathcal{U}. \end{aligned}$$

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- $X^{PRO} \subseteq X^{RO}$: set of all PRO solutions

Some questions

- Given a RO solution, is it also PRO?
- How can one find a PRO solution?
- Can we optimize over $\mathcal{X}^{\text{PRO}}$?
- Can we characterize $\mathcal{X}^{\text{PRO}}$?
  - Is it non-empty?
  - Is it convex?
  - When is $\mathcal{X}^{\text{PRO}} = \mathcal{X}^{\text{RO}}$?
- How does the notion generalize in other RO formulations?

Finding PRO solutions

Theorem

Given a solution $x \in X^{\mathrm{RO}}$ *and an arbitrary point* $\bar{p} \in \mathrm{ri}(\mathcal{U})$*, consider the following linear optimization problem:*

$$\begin{array}{ll} \text{maximize} & \bar{p}^\intercal y \\ \text{subject to} & y \in \mathcal{U}^* \\ & x + y \in \mathcal{X}. \end{array}$$

Then, either

- $\mathcal{U}^* \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : y^\intercal u \geqslant 0, \forall\, u \in \mathcal{U}\}$ is the dual of $\mathcal{U}$

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- *the optimal value is zero and* $x \in X^{\mathrm{PRO}}$*, or*

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- *the optimal value is zero and* $x \in X^{PRO}$, *or*
- *the optimal value is strictly positive and* $\bar{x} = x + y^\star \in X^{PRO}$, *for any optimal* $y^\star$.

- $\mathcal{U}^* \overset{\text{def}}{=} \{y \in \mathbb{R}^n : y^\intercal u \geqslant 0, \forall u \in \mathcal{U}\}$ is the dual of $\mathcal{U}$

Remarks

- Finding a point $\bar{u} \in \text{ri}(\mathcal{U})$ can be done efficiently using LP techniques
- Testing whether $x \in X^{\text{RO}}$ is no harder than solving the classical RO problem in this setting
- Finding a PRO solution $x \in X^{\text{PRO}}$ is no harder than solving the classical RO problem in this setting

Corollaries

- If $\bar{u} \in \mathrm{ri}(\mathcal{U})$, all optimal solutions to the problem below are PRO:

$$\begin{array}{ll} \text{maximize} & \bar{u}^\intercal x \\ \text{subject to} & x \in X^{\text{RO}} \end{array}$$

- If $0 \in \mathrm{ri}(\mathcal{U})$, then $X^{\text{PRO}} = X^{\text{RO}}$

- If $\bar{u} \in \mathrm{ri}(\mathcal{U})$, then $X^{\text{PRO}} = X^{\text{RO}}$ if and only if the optimal value of this LP is zero:

$$\begin{array}{ll} \text{maximize} & \bar{u}^\intercal y \\ \text{subject to} & x \in X^{\text{RO}} \\ & y \in \mathcal{U}^* \\ & x + y \in \mathcal{X} \end{array}$$

Optimizing over / Understanding X^{PRO}

- Secondary objective r: can we solve

$$\begin{array}{ll} \text{maximize} & r^\intercal x \\ \text{subject to} & x \in X^{\text{PRO}}? \end{array}$$

- Interesting case: $X^{\text{RO}} \neq X^{\text{PRO}}$

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Proposition

X^{PRO} *is not necessarily convex.*

- $\mathcal{X} = \{x \in \mathbb{R}^4_+ \; : \; x_1 \leqslant 1, \; x_2 + x_3 \leqslant 6, \; x_3 + x_4 \leqslant 5, \; x_2 + x_4 \leqslant 5\}$
- $\mathcal{U} = \text{conv}\Big(\big\{e_i, \; i \in \{1, \ldots, 4\}\big\}\Big)$
- $J^\star_{RO} = 1$, and $X^{RO} = \{x \in X \; : \; x \geqslant \mathbf{1}\}$
- $x^1 = \begin{bmatrix} 1 & 2 & 4 & 1 \end{bmatrix}^\intercal$, $x^2 = \begin{bmatrix} 1 & 4 & 2 & 1 \end{bmatrix}^\intercal \in X^{PRO}$
- $0.5\, x^1 + 0.5\, x^2$ is Pareto dominated by $\begin{bmatrix} 1 & 3 & 3 & 2 \end{bmatrix}^\intercal \in X^{RO}$.

Optimizing over / Understanding X^{PRO}

- Secondary objective r: can we solve

$$\begin{array}{ll} \text{maximize} & r^T x \\ \text{subject to} & x \in X^{PRO}? \end{array}$$

Proposition

If $X^{RO} \neq X^{PRO}$, *then* $X^{PRO} \cap \text{ri}(X^{RO}) = \emptyset$.

- Whether solution to nominal RO is PRO depends on algorithm used for solving LP
- Simplex **better for RO problems** than interior point methods

What Are The Gains?

Example (Portfolio)

- $n + 1$ assets, with returns r_i
- $r_i = \mu_i + \sigma_i \zeta_i$, $i = 1, \ldots, n$, $r_{n+1} = \mu_{n+1}$
- ζ unknown, $U = \{\zeta \in \mathbb{R}^n : -\mathbf{1} \leqslant \zeta \leqslant \mathbf{1},\ \mathbf{1}^\intercal \zeta = 0\}$
- Objective: select weights x to maximize worst-case portfolio return

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- ζ unknown, $\mathcal{U} = \{\zeta \in \mathbb{R}^n : -\mathbf{1} \leqslant \zeta \leqslant \mathbf{1},\ \mathbf{1}^\top \zeta = 0\}$
- Objective: select weights x to maximize worst-case portfolio return

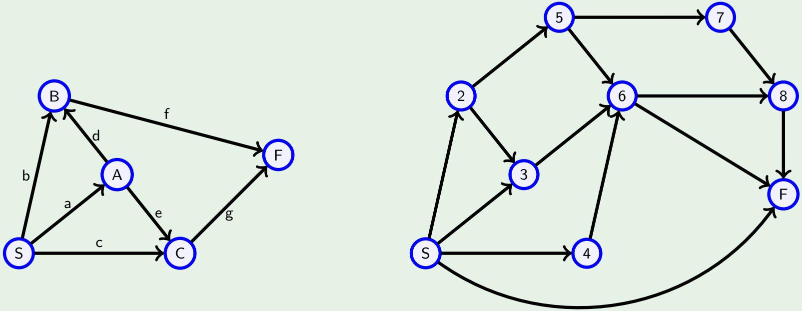
Example (Inventory)

- One warehouse, N retailers where uncertain demand is realized
- Transportation, holding costs and profit margins differ for each retailer
- Demand driven by market factors $d_i = d_i^0 + q_i^\top z, \quad i = 1, \ldots, N$
- Market factors z are uncertain

$$z \in \mathcal{U} = \{z \in \mathbb{R}^N : -b \cdot \mathbf{1} \leqslant z \leqslant b \cdot \mathbf{1},\ -B \leqslant \mathbf{1}^\top z \leqslant B\}$$

Example (Project management)

- A PERT diagram given by directed, acyclic graph $G = (\mathcal{N}, \mathcal{E})$
- $\mathcal{N}$ are project events, $\mathcal{E}$ are project activities / tasks



Example (Project management)

- A PERT diagram given by directed, acyclic graph $G = (\mathcal{N}, \mathcal{E})$
- $\mathcal{N}$ are project events, $\mathcal{E}$ are project activities / tasks
- Task $e \in \mathcal{E}$ has uncertain duration $\tau_e = \tau_e^0 + \delta_e$

$$\delta \in \mathcal{U} := \left\{\delta \in \mathbb{R}_+^{|\mathcal{E}|} : \delta \leqslant b \cdot \mathbf{1}, \quad \mathbf{1}^\intercal \delta_e \leqslant B\right\}$$

- Task $e \in \mathcal{E}$ can be expedited by allocating a budgeted resource x_e

$$\begin{aligned} \tau_e &= \tau_e^0 + \delta_e - x_e \\ \mathbf{1}^\intercal x &\leqslant C \end{aligned}$$

- Goal: find resource allocation x to minimize worst-case completion time

Results – finance and inventory examples (10K instances)

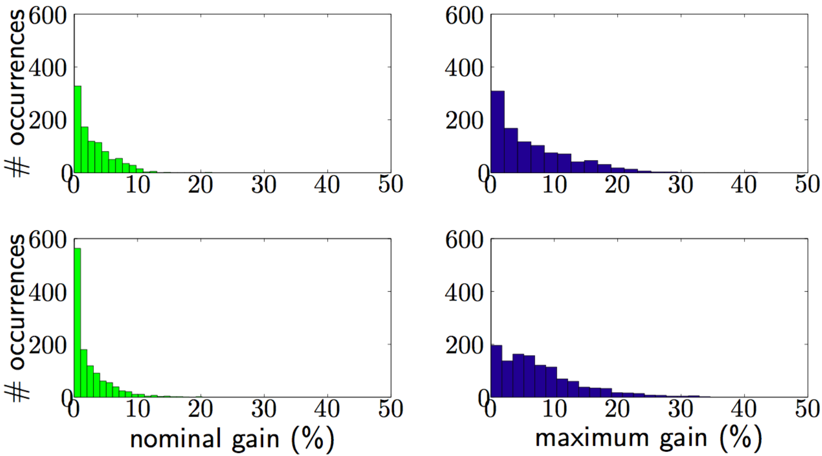
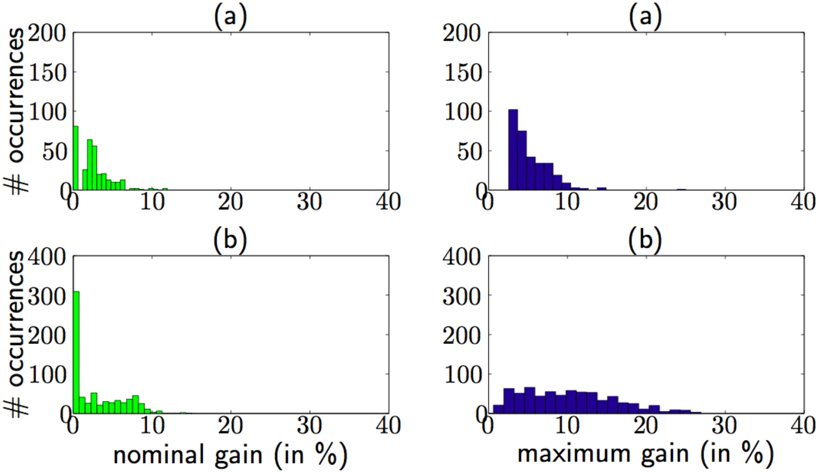


Figure: TOP: portfolio example. BOTTOM: inventory example. LEFT: performance gains in nominal scenario. RIGHT: maximal performance gains.

Results – two project management networks (10K instances)



Careful To Avoid Naïve Inefficiencies In Robust Models!

Dynamic Robust Optimization

$$x \text{ chosen} \quad \mapsto \quad z \text{ revealed} \quad \mapsto \quad y(x, z) \text{ chosen}$$

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Stochastic model:

$$\min_{x} \; \mathbb{E}_z \left[\min_{y(x,z)} f(x, y, z) \right]$$

Robust model:

$$\min_{x} \max_{z \in \mathcal{U}} \min_{y(x,z)} f(x, y, z)$$

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- Solve problems via Dynamic Programming:
  - Given $x, z \quad \to \quad$ find $y^\star(x, z) \quad \to \quad$ find $x^\star$
  - Bellman principle: $y^\star$ optimal for any given x, z

Dynamic Robust Optimization

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Outline...

1. Properly writing a robust DP
2. Tractable approximations with decision rules
3. A subtle point: is Bellman optimality really **necessary**?
   - If not, what to replace it with?
   - Why is this relevant?
4. Applications

Consider the following *deterministic* inventory management problem:

$$\begin{aligned} \underset{\{x_t\}_{t=1}^T}{\text{minimize}} \quad & \sum_{t=1}^{T} \left(\overbrace{c_t x_t}^{\text{ordering cost}} + \overbrace{h_t (y_{t+1})^+}^{\text{holding cost}} + \overbrace{b_t (-y_{t+1})^+}^{\text{backlog cost}} \right) \\ \text{s.t.} \quad & y_{t+1} = y_t + x_t - d_t, \ \forall\, t, && \text{(Stock balance)} \\ & L_t \leqslant x_t \leqslant H_t, \ \forall\, t, && \text{(Min/max order size)} \\ & y_1 = a\,, && \text{(Initial stock level)} \end{aligned}$$

A simple motivating example

Consider the following *deterministic* inventory management problem:

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where

- x_t is number of goods ordered at time t and received at $t+1$
- y_t is number of goods in stock at beginning of time t
- d_t is demand during period t
- a is the initial inventory

A simple motivating example

Consider the following *deterministic* inventory management problem:

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What if future demands known to reside in an **uncertainty set** $\mathcal{U}$?

$$\mathbf{d} := (d_1, d_2, \ldots, d_T) \in \mathcal{U} \subseteq \mathbb{R}^T$$

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Ordering policies can depend on revealed demands:
$x_t(d_{[t-1]})$, where $d_{[t-1]} := (d_1, d_2, \ldots, d_{t-1}) \in \mathbb{R}^{t-1}$.

Robust Dynamic Programming Formulation

Our dynamic decision problem can also be written:

$$\begin{aligned}
&\min_{L_1 \leqslant x_1 \leqslant H_1} \Bigg[c_1 x_1 + \max_{d_1 \in \mathcal{U}_1(\varnothing)} \Bigg[h_1(y_2)^+ + b_1(-y_2)^+ \\
&+ \min_{L_2 \leqslant x_2 \leqslant H_2} \Bigg[c_2 x_2 + \max_{d_2 \in \mathcal{U}_2(d_1)} \Bigg[h_2(y_3)^+ + b_2(-y_3)^+ + \ldots \\
&+ \min_{L_T \leqslant x_T \leqslant H_T} \Bigg[c_T x_T + \max_{d_T \in \mathcal{U}_T(d_{[T-1]})} [h_T(y_{T+1})^+ + b_T(-y_{T+1})^+] \Bigg] \ldots \Bigg]
\end{aligned}$$

where:

$$y_{t+1} := y_t + x_t - d_t$$

$$\mathcal{U}_t(d_{[t-1]}) := \left\{ d \in \mathbb{R} \, : \, \exists \, \xi \in \mathbb{R}^{T-t} \text{ such that } [d_{[t-1]};\ d\ ; \xi] \in \mathcal{U} \right\}$$

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1. Nested min-max problems
2. Explicit rule for "conditioning": *projection* of uncertainty set

Bellman Principle; Robust DP Recursions

- The **state** of the system at time t:

$$S_t := [y_t;\ d_{[t-1]}] = [y_t;\ d_1\ d_2;\ \ldots\ ; d_{t-1}] \in \mathbb{R}^T$$

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- Value function $J_t^\star(S_t)$ given by:

$$J_t^\star(S_t) = \min_{L_t \leqslant x_t \leqslant H_t} \left[c_t x_t + \max_{d_t \in \mathcal{U}_t(d_{[t-1]})} \left[h_t (y_{t+1})^+ + b_t (-y_{t+1})^+ + J_{t+1}^\star(S_{t+1}) \right] \right]$$

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Observations:

1. General $\mathcal{U} \longrightarrow$ high-dimensional $S_t \longrightarrow$ **curse of dimensionality**

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$$\mathcal{U}_{\text{box}} = \left\{\mathbf{d} : \underline{d}_t \leqslant d_t \leqslant \bar{d}_t]\right\} \quad \rightarrow \quad S_t = y_t$$

$$\mathcal{U}_{\text{budget}} = \left\{\mathbf{d} : \exists\, \boldsymbol{z}, \|\boldsymbol{z}\|_\infty \leqslant 1, \|\boldsymbol{z}\|_1 \leqslant \Gamma, d_t = \bar{d}_t + \hat{d}_t z_t\right\} \rightarrow S_t = \left[y_t, \sum_{\tau=1}^{t-1} |z_\tau|\right]^T$$

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2. When $\mathcal{U}$ has special structure, can reduce state space

- Reduce computational burden
- Prove structural results, comparative statics

$$x_t^\star(y) = \min\left(H_t, \max(L_t,\ \theta_t - y)\right) \qquad \text{(modified) } \textit{base-stock policy}$$

Back to our basic dynamic robust model:

$$\min_{x} \max_{z \in \mathcal{U}} \min_{y(z)} f(x, y, z)$$

- Finding Bellman-optimal rules $y^\star(z)$ generally intractable

Tractable Approximations Via Decision Rules

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- Ben-Tal et. al: **Linear Decision Rules**
  - Suppose we have a constraint
    $$(\bar{a} + Pz)^\intercal x + d^\intercal y(z) \leqslant b, \quad \forall z \in \mathcal{U}$$
    where $y(z)$ is dynamically adjustable
  - A linear (affine) form $y = u + Vz$ would lead to the problem:
    $$\bar{a}^\intercal x + d^\intercal u + (P^\intercal x + V^\intercal d)^\intercal z \leqslant b, \quad \forall z \in \mathcal{U}.$$
    Constraint **linear** in decisions x, u, V and uncertainty z, so all previous results apply!

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    Constraint **linear** in decisions x, u, V and uncertainty z, so all previous results apply!
- So how to apply these **static** or **linear** rules in a real problem?

Implementation and Potential Pitfalls

Recall our inventory problem. The **deterministic** version can be reformulated as an LP:

$$
\begin{aligned}
\underset{x_t, y_t, s_t^+, s_t^-}{\text{minimize}} \quad & \sum_{t=1}^{T} \left(c_t x_t + h_t s_t^+ + b_t s_t^- \right) \\
\text{s.t.} \quad & s_t^+ \geqslant 0,\ s_t^- \geqslant 0,\ \forall t, \\
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& y_{t+1} = y_t + x_t - d_t,\ \forall t, \\
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where

- s_t^+ : physical inventory held at end of period t
- s_t^- : backlogged customer demand at end of period t

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Naïve Robustification

Consider a naïve robust optimization model:

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$$

Unfortunately, this is **infeasible** even when $\mathcal{U} = \{\mathbf{d}^{(1)}, \mathbf{d}^{(2)}\}$:

$$
\left\{ \begin{array}{l} y_{t+1} = y_t + x_t - d_t^{(1)} \\ y_{t+1} = y_t + x_t - d_t^{(2)} \end{array} \right\} \Rightarrow\ d_t^{(1)} = d_t^{(2)}
$$

Problem arises due to "$=$" constraint!

A less naïve robustification

Robustify an alternate linear programming formulation:

$$\begin{aligned}
\underset{x_t, s_t^+, s_t^-}{\text{minimize}} \quad & \sum_t \left(c_t x_t + h_t s_t^+ + b_t s_t^- \right) \\
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where we simply replace $y_{t+1} := y_1 + \sum_{t'=1}^{T} (x_{t'} - d_{t'})$.

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Q: If orders x_t are **static** (i.e., fixed $t = 0$), should (s_t^+, s_t^-) also be static?

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Q: If orders x_t are **static** (i.e., fixed $t = 0$), should (s_t^+, s_t^-) also be static?

A: No, that would be **unnecessarily conservative**!
Auxiliary (i.e., "reformulation") variables should be fully adjustable, even under static "implementable" decisions.

Linear Decision Rules

- Take both **ordering policies** and **auxiliary variables** to depend *linearly* on demands

$$x_t\left(d_{[t-1]}\right) = x_t^0 + X_t d_{[t-1]}$$

$$s_t^+\left(d_{[t-1]}\right) = s_t^+ + S_t^+ d_{[t-1]}$$

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- The **Robust Counterpart** problem becomes:

$$\begin{aligned}\min_{\mathcal{X}} \quad & \max_{d \in \mathcal{U}} \sum_{t=1}^{T} c_t \cdot (x_t^0 + X_t d) + h_t \cdot (s_t^+ + S_t^+ d) + b_t \cdot (s_t^- + S_t^- d) \\ \text{s.t.} \quad & s_t^+ + S_t^+ d \geqslant 0, \quad s_t^- + S_t^- d \geqslant 0, \ \forall\, d \in \mathcal{U} \\ & s_t^+ + S_t^+ d \geqslant y_1 + \sum_{\tau=1}^{T} (x_\tau^0 + X_\tau d_{[\tau-1]} - d_\tau), \ \forall\, d \in \mathcal{U}, \\ & s_t^- + S_t^- d \geqslant -y_1 - \sum_{\tau=1}^{T} (x_\tau^0 + X_\tau d_{[\tau-1]} - d_\tau), \ \forall\, d \in \mathcal{U}, \\ & L_t \leqslant x_t + X_t d \leqslant H_t \,, \ \forall\, d \in \mathcal{U},\end{aligned}$$

- **Decision variables:** coefficients $\mathcal{X} = \left\{x_t^0, X_t, s_t^+, S_t^+, s_t^-, S_t^-\right\}_{t=1}^{T}$
- **Two** layers of sub-optimality: **policies** *and* **auxiliary variables**; any good?

Empirical Performance: Ben-Tal et al. ('04, '09), with **box** uncertainty

ρ (%)	**OPT**	**Linear (Gap)**	**Static (Gap)**
10	13531.8	13531.8 (+0.0%)	15033.4 (+11.1%)
20	15063.5	15063.5 (+0.0%)	18066.7 (+19.9%)
30	16595.3	16595.3 (+0.0%)	21100.0 (+27.1%)
40	18127.0	18127.0 (+0.0%)	24300.0 (+34.1%)
50	19658.7	19658.7 (+0.0%)	27500.0 (+39.9%)
60	21190.5	21190.5 (+0.0%)	30700.0 (+44.9%)
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Theorem (Bertsimas, I., Parrilo 2010, I., Sharma & Sviridenko 2013)

*For any **convex** order costs $c_t(\cdot)$ and inventory costs $h_t(\cdot)$, affine orders $x_t(\mathbf{d}_{[t-1]})$ and affine auxiliary variables $s_t^{+,-}(\mathbf{d}_{[t-1]})$ generate the optimal worst-case cost.*

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Why is this relevant?

1. *Insight*: orders only depend on **backlogged** demand
2. *Computational*: if c_t, h_t piecewise affine (m pieces), must solve an LP of $\mathcal{O}(m \cdot T^2)$.
3. *Extensions*: can embed decisions at $t = 0$ (e.g., capacities, order pre-commitments)
4. *Robust dynamic* **critically different** *from stochastic dynamic*
   - Stochastic model with complete $\mathbb{P}$ requires "complex" policies; affine very suboptimal
   - Robust model admits a very "simple" class of optimal policies

A Critical Difference: Bellman Optimality in Stochastic and Robust Models

$$\text{"Nature" reveals } z \quad \mapsto \quad \text{DM chooses } y(z)$$

Stochastic model:

$$J^\star_{\text{sto}} = \mathbb{E}_z\Big[\min_{y(z)} \; f(y, z)\Big]$$

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$$\mathcal{Y}^{\text{wc}} := \big\{ y : \mathcal{U} \to \mathbb{R}^m \;:\; f\big(y(z), z\big) \leqslant J^\star_{\text{rob}}, \; \forall\, z \in \mathcal{U} \big\}.$$

  will be "optimal" in the robust problem, i.e., $\max_{z\in\mathcal{U}} f(y^{\text{wc}}(z), z) = J^\star_{\text{rob}}$

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- The set of **worst-case optimal policies** $\mathcal{Y}^{\text{wc}}$ is non-empty and degenerate
  - There are **infinitely many** worst-case optimal policies

Implications for Robust Dynamic Models

1. Bellman optimality **not necessary**; **worst-case optimality** necessary
   - Introduces *degeneracy* in policies/decisions

2. This degeneracy is typical for robust multi-stage problems
   (“If adversary does not play optimally, you don’t have to, either...”)

3. Critically different from stochastic problems

4. A blessing: may allow finding policies with **simple structure**
   - e.g., affine...

5. A curse: may yield Pareto inefficiencies in the decision process
   - I. and Trichakis [2014] discuss a potential fix

6. Worst-case optimal policies **must be implemented with resolving**

A Monitoring Problem

Disease Monitoring

Debt Monitoring

- **Significant uncertainty**
  - limited data to calibrate dynamic evolution
- **System can be monitored at finite number of times**
  - *e.g., healthcare*: testing requires office visit, expensive/invasive procedures
  - *e.g., (micro-)lending*: on-site visits, costly appraisals of collateral, etc.
  - **monitoring times must be chosen judiciously**
- **Complex, high-dimensional decision problem**
  - processes influence each other; monitoring / learning adds complexity

Robust Monitoring and Stopping [I., Trichakis, Yoon]

- Consider a system evolving over continuous time $[0, T]$
- State characterized by d processes, denoted $x(t) \in \mathbb{R}^d$
- A decision maker (DM) starts with initial information $x(0)$
- Can monitor the system at most n times, at $0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T$. (let $t_0 \stackrel{\text{def}}{=} 0, t_{n+1} \stackrel{\text{def}}{=} T$)
- At each time t_p, the DM:
  - Observes the state $x_p \stackrel{\text{def}}{=} x(t_p)$
  - Updates information about possible future state values
  - Decides whether to stop or not
- When stopping at t, collect $g(t, x(t))$

Uncertainty Sets and Information Updating

- An observation $\boldsymbol{x}_p$ at t_p imposes restrictions on future state values $\boldsymbol{x}(t)$ (for $t > t_p$) summarized through m constraints:

$$f(t_p, t, \boldsymbol{x}_p, \boldsymbol{x}(t)) \leqslant 0$$

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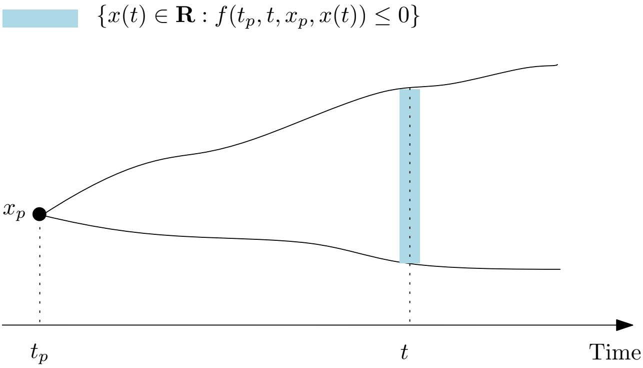

Case $d = 1$. Information about $x(t)$ acquired at $t_p < t$.

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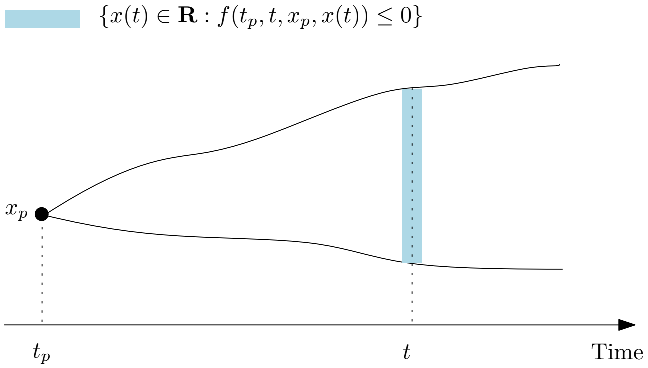

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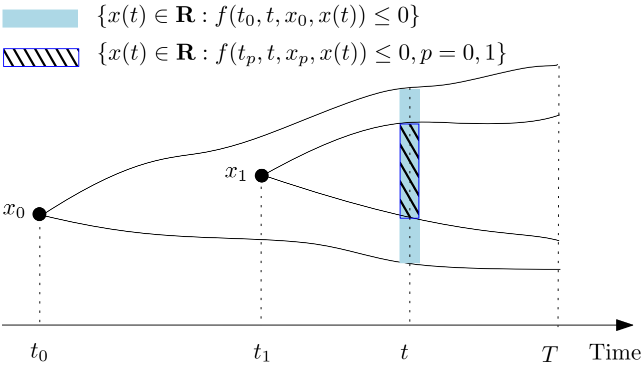

Case $d = 1$. Information about $x(t)$ acquired at t_0 and t_1.

Uncertainty Sets and Information (Updating)

- Suppose DM committed to r monitoring times: $\mathbf{t}^{\{r\}} = [t_0, t_1, \ldots, t_r]$
- DM made $k \leqslant r$ observations so far: $\boldsymbol{x}^{\{k\}} = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$.
- The future possible process values at times $t_{k+1}, \ldots, t_r, T$ lie in:

$$\mathcal{U}\left(\mathbf{t}^{\{r\}}, \boldsymbol{x}^{\{k\}}\right) = \Big\{[\boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_r, \boldsymbol{x}_{n+1}] \in \mathbb{R}^{d \times (r-k+1)} : \\ f(t_p, t_q, \boldsymbol{x}_p, \boldsymbol{x}_q) \leqslant 0, \forall\, p, q \in \{0, 1, \ldots, r, n+1\}, p < q\Big\},$$

where $\mathbf{t}^{\{r\}} = [t_0, t_1, \ldots, t_r]$ and

- **Notation.** Let $\mathcal{U}_{k+1}$ = set of possible values for $\boldsymbol{x}_{k+1}$ (by projecting $\mathcal{U}$ above)

DM's Monitoring and Stopping Problem

- **Goal:** Find monitoring and stopping policy to maximize worst-case reward
  - Two versions, depending on choice of monitoring times

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$$V_k(\mathbf{t}^{\{n+1\}}, \boldsymbol{x}^{\{k\}}) = \max\Big(\underbrace{g\big(t_k, x_k\big)}_{\text{stop}}, \underbrace{\min_{x_{k+1} \in \mathcal{U}_{k+1}(\mathbf{t}^{\{n+1\}}, \boldsymbol{x}^{\{k\}})} V_{k+1}\big(\mathbf{t}^{\{n+1\}}, \boldsymbol{x}^{\{k+1\}}\big)}_{\text{continue}}\Big),$$

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  - DM seeks a **monitoring policy**: $\tau_k^D(\mathbf{t}^{\{k\}}, \boldsymbol{x}^{\{k\}})$.

Assumptions

Assumption (Monotonic Rewards)

$g(t, x)$ *component-wise monotonic in* x.

- Each state is either “good” or “bad”

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Assumption (U-set Structure)

For any $0 \leqslant k \leqslant r \leqslant n$ *and given* $\mathbf{t}^{\{r\}}$ *and* $\mathbf{x}^{\{k\}}$,

i. **(Lattice)** $\mathcal{U}\left(\mathbf{t}^{\{r\}}, \mathbf{x}^{\{k\}}\right)$ *is a lattice;*

ii. **(Monotonicity)** $\mathcal{U}\left(\mathbf{t}^{\{r\}}, \mathbf{x}^{\{k\}}\right)$ *is increasing in* $\mathbf{x}^{\{k\}}$;

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- Lattice: technical requirement
- Monotonicity: better past $\rightarrow$ better future
- Consistency: future monitoring times $t_{k+2}, \ldots, t_r$ do not impact possible values for x_{k+1}

Examples

Example (Lattice with Cross-Constraints)

For $\mathcal{M} \subseteq \{1, \dots, d\}^2$, $\ell : \mathbb{R}^2 \to \mathbb{R}_-$ decreasing in its second argument, and $u : \mathbb{R}^2 \to \mathbb{R}_+$ increasing in its second argument:

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Example (CLT-Budgeted Uncertainty Sets)

For $\Gamma > 0$, $\sigma > 0$, and μ,

$$\mathcal{U}(\mathbf{t}^{\{r\}}, \mathbf{x}^{\{k\}}) = \Big\{[x_{k+1}, \ldots, x_r, x_{n+1}] \in \mathbb{R}^{r-k+1} : \\ -\Gamma \leqslant \frac{x_q - x_p - (t_q - t_p)\mu}{\sqrt{t_q - t_p}\,\sigma} \leqslant \Gamma, \forall p, q \in \{0, \ldots, r, n+1\},\ p < q\Big\}.$$

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Theorem (Solving Static Problem)

The worst-case optimal value under static monitoring can be obtained as:

$$\max_{\mathbf{t}^{\{n+1\}}} \max_{k \in \{n, n+1\}} g\big(t_k, \underline{x}_k(\mathbf{t}^{\{n+1\}})\big).$$

- Without loss, can choose times to either stop at t_n or at T
- $\underline{x}_k(\mathbf{t}^{\{n+1\}})$ is the worst-case scenario (smallest state under $\mathbf{t}^{\{n+1\}}$)

Stationary Uncertainty Sets

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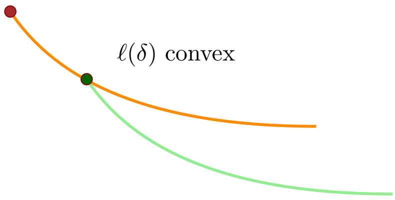

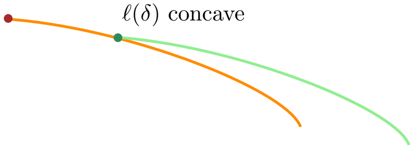

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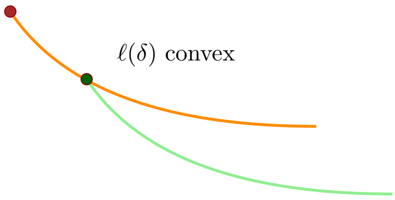

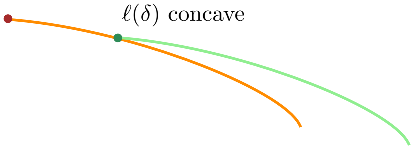

- Worst-case optimal to **stop at** t_1
  - $t_2, \ldots, t_n$ redundant
- Find t_1: solve 1D optimization

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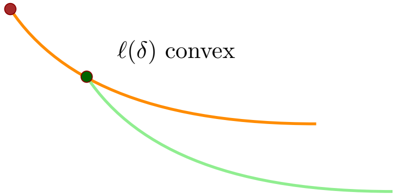

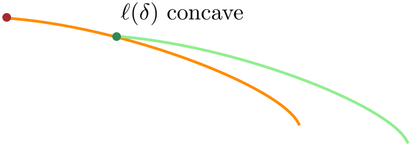

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- Worst-case optimal to either:
  - Stop at T; monitor at $t_k^\star = \frac{kT}{n+1}$

    or
  - Stop at t_n; monitor at $t_k^\star \frac{k\,t_n}{n}$
- Find t_n: solve 1D optimization
- **Uniform-interval** monitoring optimal!

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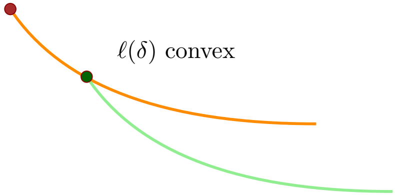

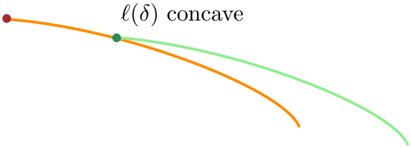

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- Paper also has additional results on general case ...
- ... and a case study in Cardiac Allograft Vasculopathy

Conclusions

- Robust decision making: a very relevant/realistic framework
  - *"When do we really have complete probabilistic descriptions?"*
- A powerful framework
  - **Flexible:** allows embedded various levels of information
  - **Tractable:** can solve many classes of problems (if suitable formulated)
  - **Not necessarily conservative** (if suitable formulated)
- Very useful (theory and practice)
  - theory: can be used to rationalize simple rules that work well
- Has a few "quirks"
  - Careful with formulating nominal model
  - Careful to avoid inefficiencies
- Specific areas in OM where it could be used more:
  - contracting in complex value chains (developing world, disruption risk, ...)
  - behavioral operations

References I

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A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming*. MPS / SIAM Series on Optimization. SIAM, 2009.