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Lectures 7-8: Integer Programming

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In this lecture, we consider the following type of optimization problems:

$$\begin{aligned} \min\ & c^\intercal x + d^\intercal y \\ & Ax + By = b \\ & x, y \geq 0 \\ & x \text{ integer} \end{aligned}$$

We call this problem a **mixed integer programming** problem. If there are no continuous problem, then we simply call this an integer program (IP). Moreover, if x is further constrained to take value in $\{0,1\}^n$, we call this a zero-one or **binary** optimization problem. As we will soon see, these problems offer a very powerful modeling framework, but the downside is that they are generally hard to solve to optimality. We then discuss a few special cases when these problems are solvable (essentially, as easy as LPs) and we highlight some of the algorithms used to tackle these problems in full generality.

1 Modeling Techniques

Integer programming offers a very rich modeling framework. Here are some examples to illustrate this.

1.1 Binary choice

A binary variable can represent one of two alternatives. For instance, consider the classical **knapsack problem**.

Example 1 (The zero-one knapsack problem)**.** *We are given n items. The j-th item has weight w_j and its value/reward is r_j. Given a bound K on the weight that can be carried in a knapsack, we would like to select items to maximize the total value.*

To model this problem, we define a binary variable x_j which is 1 if item j is chosen, and 0 otherwise. The problem can then be formulated as follows:

$$\begin{aligned} \text{maximize } & \sum_{j=1}^{n} r_j x_j \\ \text{subject to } & \sum_{j=1}^{n} w_j x_j \leq K \\ & x_j \in \{0,1\}, \quad j = 1, \ldots, n. \end{aligned}$$

1.2 Logical constraints

We already saw several examples of logical constraints in our third class. Here, we briefly remind you of some of the basic building blocks. Suppose that we have activities/projects A and B and we use binary variables with the same name to indicate whether each activity is conducted; so $A = 1$ if and only if activity A is done. Then:

- to impose the condition: "if activity A is done, then activity B should also be done," we should add the constraint $A \leq B$. This exactly implements the **logical "or"** between the two projects: note that the condition that A or B should be done means $A + B \geq 1$, which is exactly equivalent to our constraint.
- To implement the logical **not**, we can use $1 - A$. That is, A **is not done** if and only if $1 - A = 1$.
- To implement the logical **and** – for instance, to create the binary variable $Z = A \cdot B$ – we can add the constraints:

$$Z \leq X,\ Z \leq Y,\ Z \geq X + Y - 1.$$

Moreover, if x is an n-dimensional vector of continuous or discrete decisions and $a \in \mathbb{R}^n, b \in \mathbb{R}$, to implement the condition that

$$Y = 1 \quad \Leftrightarrow \quad a^\intercal x + b \geq 0,$$

we should add the two constraints:

$$a^\intercal x + b \geq m \cdot (1–Y)$$
$$a^\intercal x + b + \epsilon \leq (M + \epsilon) \cdot Y,$$

where m and M are the smallest and largest value, respectively, that $a^\intercal x + b$ can take over any feasible x, and ϵ is a very small tolerance parameter. The parameter arises because with continuous variables x, it is impossible to enforce precisely a strict inequality. However, if $x \in \mathbb{Z}^n$, the epsilon can be replaced with a finite value to obtain an exact reformulation.

Facility Location. As a classical example of logical constraints in practice, consider the facility location problem where we have n potential locations and m clients who need service. There is a fixed cost c_j for opening a facility at location j and a cost d_{ij} for serving client i from facility j. The goal is to select a set of facility locations and assign each client to one of the facilities at minimum cost. (For a visualization, see Figure 1.)

$$\begin{aligned}
\min\ & \sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij} x_{ij} \\
& \sum_{j=1}^{n} x_{ij} = 1 \\
& x_{ij} \leq y_j \\
& x_{ij}, y_j \in \{0, 1\}
\end{aligned}$$

Here, the equality constraint $\sum_{j=1}^{n} x_{ij} = 1$ model the requirement that each client i is exactly matched with one of the facilities j, and $x_{ij} \leq y_j$ ensures that clients can only be matched with open facilities.

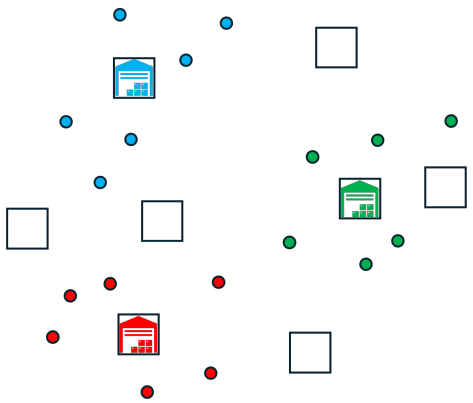


Figure 1: A facility location problem. Black squares denote potential locations for facilities, circles denote customers, and the cost function is Euclidean distance. The color-coding denotes the match between the three open facility and the customers.

1.3 Restricted range of values

Suppose we need to restrict a variable x to take values in a set $\{a_1, \ldots, a_m\}$. This can be achieved with the following constraints:

$$x = \sum_{j=1}^{m} a_j y_j, \quad \sum_{j=1}^{m} y_j = 1, \quad y_j \in \{0, 1\}.$$

1.4 Arbitrary piecewise linear cost functions

Binary variables allow reformulating an arbitrary piecewise-linear cost function. Suppose that $a_1 < a_2 < \cdots < a_k$ and that we have a continuous[1] piecewise linear function $f(x)$ specified by the points $(a_i, f(a_i))$ for $i = 1, \ldots, k$, defined on the interval $[a_1, a_k]$ (see Figure 2). Then, any $x \in [a_1, a_k]$ can be expressed in the form

$$x = \sum_{i=1}^{k} \lambda_i a_i,$$

where $\lambda_1, \ldots, \lambda_k$ are nonnegative scalars that sum to one.

Importantly, although the choice of coefficients $\lambda_1, \ldots, \lambda_k$ used to represent a particular x is not unique, this becomes unique if we require that at most two consecutive coefficients λ_i can be nonzero. In this case, any $x \in [a_i, a_{i+1}]$ is represented uniquely as $x = \lambda_i a_i + \lambda_{i+1} a_{i+1}$, with $\lambda_i + \lambda_{i+1} = 1$, and

$$f(x) = \sum_{i=1}^{k} \lambda_i f(a_i).$$

We also need to model the additional constraint that at most two consecutive coefficients λ_i are nonzero. To this effect, we consider a binary variable y_i, $i = 1, \ldots, k-1$, which can be

[1] Discontinuous functions can also be readily accommodated by introducing additional constraints.

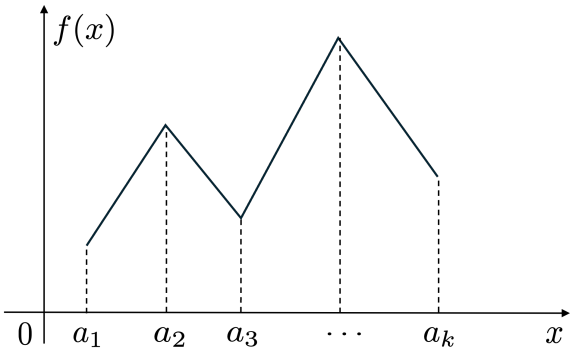


Figure 2: A piecewise linear cost function.

equal to 1 only if $a_i \leq x \leq a_{i+1}$, and must be 0 otherwise. The problem is then formulated as the following mixed integer programming problem:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{k} \lambda_i f(a_i) \\
\text{subject to} \quad & \sum_{i=1}^{k} \lambda_i = 1, \\
& \lambda_1 \leq y_1, \\
& \lambda_i \leq y_{i-1} + y_i, \quad \forall i = 2, \ldots, k-1, \\
& \lambda_k \leq y_{k-1}, \\
& \sum_{i=1}^{k-1} y_i = 1, \\
& \lambda_i \geq 0, \quad y_i \in \{0, 1\}, \quad \forall i.
\end{aligned}
$$

Notice that if $y_j = 1$, then $\lambda_i = 0$ for i different than j or $j+1$.

These constraints are so important in practice that they bear a special name: ***special ordered sets (SOS) of type 2)****. When adding an SOS constraint of type 2, you just need to specify a list of non-negative decision variables of which at most two can be non-zero and these must be consecutive in their ordering. (In our formulation above, these would be the ordered list of λ variables, λ_1, λ_2, etc.) Several modeling languages support adding such constraints directly and would handle the addition of any necessary binary variables and constraints "behind the scenes." For instance, in Gurobipy, you can add an SOS constraint with the syntax* `Model.addSOS ( type, vars, weights=None )` *where* `type` *specifies the type of SOS constraint (*`GRB.SOS_TYPE2` *for type 2 SOS). In case you are wondering, a type 1 SOS constraint specifies that exactly one variable from a given list can be nonzero. For more details, you can read here.*

1.5 Set covering, set packing, and set partitioning problems

Consider a set of ground objects $M = \{1, \ldots, m\}$ and let $M_1, M_2, \ldots, M_n$ be a given collection of subsets of M. We are also given a weight c_j for each set M_j in the collection. (Depending on the application, we may want more or less weight!)

Set covering. In the **set covering** problem, we seek a collection of sets M_j so that their union includes (i.e., **covers**) M and has minimum weight. To capture this mathematically, define an **incidence matrix** A with one row for each ground object $i = 1, \ldots, m$ and one column for each set $M_j, j = 1, \ldots, n$ and such that $A_{i,j} = 1$ if $i \in M_j$ and $A_{i,j} = 0$ otherwise. The set cover problem is then:

$$\begin{aligned} \text{minimize}_x \ & w^\intercal x \\ & Ax \geq e \\ & x \in \{0,1\}^n. \end{aligned}$$

The facility location example is a type of covering problem (customers must be covered from the open locations) and the ambulance placement problem you saw in the first homework is another example. There are many others in practice: crew scheduling in public transportation (where the elements of M are specific shifts or bus routes to cover and the sets M_j denote the ability/availability of each driver j), sensor placement (elements are locations that require sensing and each set M_j corresponds to a sensor placement choice that covers some locations), etc.

Set packing. In the **set packing** problem, we try to include as many disjoint sets M_j as possible in order to maximize the weight of the included elements. Mathematically, with the same incidence matrix A as above, the set packing problem is then:

$$\begin{aligned} \text{maximize}_x \ & w^\intercal x \\ & Ax \leq e \\ & x \in \{0,1\}^n. \end{aligned}$$

Again, there are many practical examples. Consider flight crew scheduling, where an airline needs to assign flight crews to flights, but a crew cannot be assigned to overlapping flight schedules (e.g., two flights departing at the same time). The elements $i \in M$ are the flights and each set M_j represents the flights that a crew can cover based on availability. For another example, consider a logistics company that needs to allocate containers to ships, but each ship has limited capacity. Different shipments may overlap in terms of size and weight, and only disjoint combinations of shipments can be placed on a single ship. The elements $i \in M$ would be the shipments that need to be loaded, and each set M_j represents a collection of shipments that fit within the capacity of a ship without overlapping in size and weight. (The airline revenue management problem was also a type of packing problem!)

Set partitioning. In a **set partitioning** problem, we seek sets M_j that form a partition of M, i.e., they are disjoint and they cover M. Both maximization and minimization versions

are possible here:

$$\begin{aligned} \text{maximize}_x \ & w^\intercal x \\ & Ax = e \\ & x \in \{0,1\}^n. \end{aligned}$$

1.6 Matching Problems

Matching problems are among the most ubiquitous in practice: riders being matched with drivers (in ride-sharing platforms), patients awaiting for a transplant being matched with an organ available for transplantation, etc.

To formulate a matching problem, consider a set U of tasks that must be completed and a set V of persons available to complete the tasks. Each task can be assigned to at most one person and each person is only able to complete only some of the tasks (e.g., due to skills). If task $i \in U$ is assigned to person $j \in V$, there is a reward of w_{ij}. A matching is an assignment of tasks to persons so that each task is done by at most one person and each person works on at most one task, and the goal is to find a matching that maximizes the total reward. We represent the matching abstractly through an undirected, bipartite graph $G = (\mathcal{N}, \mathcal{E})$ where the set of nodes $\mathcal{N}$ is partitioned into the two sets U, V ($\mathcal{N} = U \cup V, U \cap V = \emptyset$), nodes $i \in U$ denote tasks, nodes $j \in V$ denote persons, and an edge $\{i, j\} \in \mathcal{E}$ with $i \in U$ and $j \in V$ indicates that j is able to complete task i. (See Figure ?? for a visualization.)

ìcs[width=0.3]Discrete/Figures/max$_w eight_m atching.png$

Figure 3: A maximum weight matching problem.

With decisions $x_e \in \{0, 1\}$ denoting whether edge $e = \{i, j\}$ is selected – meaning task i is assigned to person j – the maximum weight matching problem is:

$$\begin{aligned} \text{maximize} \ & \sum_{e \in E} w_e x_e \\ \text{subject to} \ & \sum_{e \in \delta(i)} x_e \leq 1, \quad \forall\, i \in N, \\ & x_e \in \{0, 1\}, \end{aligned}$$

where we use the notation $\delta(i) := \{j : \{i, j\} \in \mathcal{E}\}$ to capture all nodes j adjacent to node i.

Other variations of this problem are possible. For instance, you may encounter matching problems that involve minimizing a cost and subject to a constraint that one side has to be matched fully (e.g., all the jobs must be completed), in which case the constraints would become $\sum_{e \in \delta(i)} x_e \geq 1, \forall\, i \in U$. It is also common to consider a **perfect matching**, which is one where there is no unmatched node in the graph. (This is only possible in bipartite graphs with $|U| = |V|$, and then the constraints would read $\sum_{e \in \delta(i)} x_e = 1$ for any i.) Lastly, matching problems can be formulated in more general graphs rather than the bipartite examples we considered.

1.7 The minimum spanning tree problem

Let $G = (\mathcal{N}, \mathcal{E})$ be an undirected graph with node set $\mathcal{N}$ ($|\mathcal{N}| = n$) and edge set $\mathcal{E}$ ($|\mathcal{E}| = m$). Every edge $e \in \mathcal{E}$ has an associated cost c_e. We consider the problem of finding the **minimum spanning tree** (MST), i.e., a subset of the edges that connect all the nodes in $\mathcal{N}$ at minimum cost. To formulate the problem, we define a variable x_e for each $e \in \mathcal{E}$ that is equal to 1 if edge e is included in the tree and zero otherwise.

For this problem we actually consider two distinct formulations. The first is based on the idea that a spanning tree on n nodes should be a **connected graph containing** $n-1$ **edges**. To have $n-1$ edges, the following constraint must be satisfied:

$$\sum_{e \in \mathcal{E}} x_e = n - 1.$$

For the tree to be connected, any subset of nodes $S \subset \mathcal{N} (S \neq \emptyset)$ should be connected with nodes in $\mathcal{N} \setminus S$ through at least one edge. So if we define the **cutset** $\delta(S)$:

$$\delta(S) := \big\{\{i,j\} : i \in S, j \notin S\big\}, \tag{1}$$

we can provide the following **cutset formulation** for the MST problem:

$$\textbf{(Cutset MST)} \qquad \begin{aligned} \text{minimize } & \sum_{e \in \mathcal{E}} c_e x_e \\ & \sum_{e \in \mathcal{E}} x_e = n - 1, \\ & \sum_{e \in \delta(S)} x_e \geq 1, \quad S \subset N, S \neq \emptyset \\ & x_e \in \{0,1\}. \end{aligned} \tag{2}$$

Note that the cutset formulation involves an exponential number of constraints, one for each subset $S \subset \mathcal{N}, S \neq \emptyset$.

An alternative – and equivalent – definition of a tree is based on the idea that a tree on n nodes should have exactly $n-1$ edges and no cycles. It can be shown that the tree is guaranteed to not contain a cycle if for any nonempty set $S \subset \mathcal{N}$, the number of edges with both endpoints in S is less than or equal to $|S| - 1$. For any $S \subset \mathcal{N}$, we define

$$\mathcal{E}(S) = \big\{\{i,j\} \in \mathcal{E} : i, j \in S\big\}, \tag{3}$$

and we can express these constraints as:

$$\sum_{e \in \mathcal{E}(S)} x_e \leq |S| - 1, \quad S \subset N, S \neq \emptyset, N.$$

This leads to the following IP formulation of the MST problem:

$$\textbf{(Subtour-elimination MST)} \qquad \begin{aligned} \text{minimize} \quad & \sum_{e\in\mathcal{E}} c_e x_e \\ & \sum_{e\in\mathcal{E}} x_e = n-1, \\ & \sum_{e\in\mathcal{E}(S)} x_e \le |S|-1, \quad S\subset N, S\neq\emptyset, N, \\ & x_e \in \{0,1\}. \end{aligned} \tag{4}$$

This is called the **subtour elimination formulation** because it contains constraints that eliminate all subtours (cycles over subsets of vertices). Note that this also involves an exponential number of constraints.

The two formulations – cutset and subtour elimination – can be visualized in Figure 4.

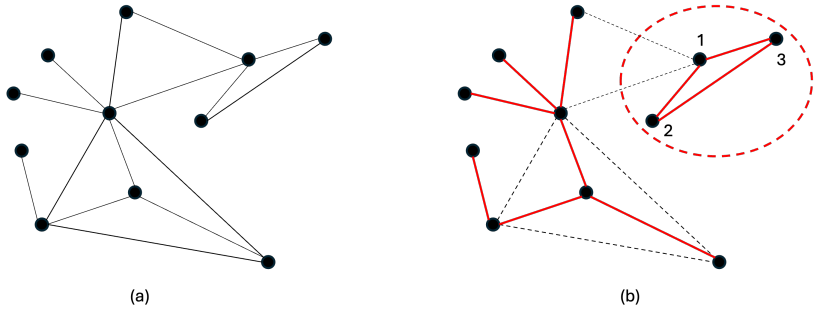


Figure 4: Formulation for the Minimum Spanning Tree Problem. The initial graph $G = (\mathcal{N}, \mathcal{E})$ is depicted in (a). Panel (b) shows a choice of edges that satisfies $\sum_{e\in\mathcal{E}} x_e = n-1$ but is not a valid tree. Note that the **cutset** formulation would rule this out because the subset of nodes $S = \{1,2,3\}$ is not connected with nodes $\mathcal{N}\setminus S$, i.e., $\delta(S) = 0$. The **subtour elimination** formulation would also rule this out because $\sum_{e\in\mathcal{E}(S)} x_e = |S| > |S|-1$.

1.8 Traveling salesperson problem

Given an undirected graph $G = (\mathcal{N}, \mathcal{E})$ and cost c_e for each edge, the objective is to find a **tour** (i.e. a cycle that visits each node exactly once) with minimum cost. To model this problem, we again use a variable x_e to denote whether an edge belongs to the tour. Mirroring the MST problem, the TSP also admits two formulations – a **cutset formulation** and a **subtour elimination** formulation – as follows.

$$\textbf{(Cutset TSP)} \qquad \begin{aligned} \text{minimize} \quad & \sum_{e\in} c_e x_e \\ & \sum_{e\in\delta(\{i\})} x_e = 2, \forall i \in N \\ & \sum_{e\in\delta(S)} x_e \ge 2, \forall S\subset N, S\neq\emptyset. \end{aligned} \tag{5}$$

Note that this is slightly different than the cutset MST formulation. In the cutset TSP formulation, any node i should have **exactly one edge coming into it and one edge leaving it** and any nontrivial subset of nodes S ($S \neq \emptyset, N$) should have **at least two edges** joining S with $Nset \setminus S$. This is because in TSP, we are interested in a tour, whereas in the MST we wanted a tree (which should be free of tours!)

The following formulation is also valid for the TSP (we omit the objective):

$$\textbf{(Subtour-elimination TSP)} \qquad \begin{aligned} \sum_{e \in \delta(\{i\})} x_e &= 2, \forall i \in N \\ \sum_{e \in \mathcal{E}(S)} x_e &\leq |S| - 1, \forall S \subset N, S \neq \emptyset \end{aligned} \tag{6}$$

The key difference with the MST formulation lies again in the first set of constraints.

The two formulations are depicted in Figure 5.

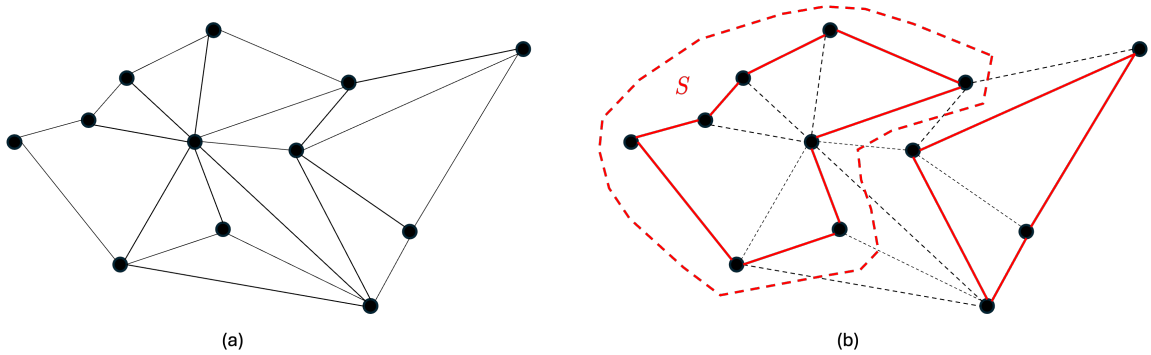


Figure 5: Formulation for Traveling Salesman Problem (TSP). The initial graph $G = (\mathcal{N}, \mathcal{E})$ is depicted in (a). Panel (b) shows a choice of edges that satisfies $\sum_{e \in \delta(i)} x_e = 2$ for any $i \in \mathcal{N}$, but is not a valid tour. Note that the **cutset** formulation rules this out because the subset of nodes $S = \{1, 2, 3\}$ is not connected with nodes $\mathcal{N} \setminus S$, i.e., $\delta(S) = 0$. The **subtour elimination** formulation would also rule this out because $\sum_{e \in \mathcal{E}(S)} x_e = |S| > |S| - 1$.

2 The Bad News First

Unfortunately, linear optimization **over integers** is **significantly harder** than over continuous variables. The following examples illustrate some of the challenges.

Example 2 (Solution Not Attained)**.** *Consider the optimization problem:*

$$\begin{aligned} \sup_{x,y} \; & x + \sqrt{2}y \\ & x + \sqrt{2}y \leq \frac{1}{2} \\ & x, y \in \mathbb{Z}. \end{aligned}$$

The optimal value is not attained.

You can probably quickly see that the optimal value in this problem is $\frac{1}{2}$ and it would be achieved with any choice of x and y such that $x + \sqrt{2}y = \frac{1}{2}$. But unfortunately, no integer values of x, y would ever satisfy this with equality. Note that this problem would never arise with continuous x, y, where the optimal value would be trivially achieved.

Example 3 (No Strong Duality)**.** *Consider the following pair of optimization programs:*

$$(\mathscr{P}) \quad \min\ x \qquad\qquad (\mathscr{D}) \quad \max_{p}\ p$$
$$2x = 1 \qquad\qquad 2p \leq 1$$
$$x \geq 0$$

With $x \in \mathbb{R}$ and $p \in \mathbb{R}$, the problems constitute a primal-dual pair; both are feasible and the optimal value (for each) is $\frac{1}{2}$. With $x \in \mathbb{Z}$ and $p \in \mathbb{Z}$, problem $(\mathscr{P})$ does not have any feasible solution, but problem $(\mathscr{D})$ is feasible and has optimal value 0.

This example shows that strong duality fails with discrete variables: we have an optimization problem that has a finite optimal value (the dual $(\mathscr{D})$) but its dual is infeasible. (It is easy to construct examples where the mirroring situation also happens, i.e., the primal minimization has a finite optimal value but the dual maximization problem is infeasible).

In fact, IPs are – in theory and practice – significantly more difficult than LPs.

Theorem 1. *Given a matrix $A \in \mathbb{Q}^{m \times n}$ and a vector $b \in \mathbb{Q}^m$, the problem: "does $Ax \leq b$ have an integral solution x" is **NP-complete**.*

The theorem states that the "feasibility problem" in integer programming is already NP-complete, which means it is the hardest type of problem in NP. (We will not be discussing complexity results too much in this class, but as a quick reminder, problems in NP are problems that admit a polynomial-time verification of a YES instance. For instance, in our IP feasibility problem, if we are given an x that is actually feasible, it is easy to verify whether it works – we just need to check the constraints!) For a proof of the result, see Theorem 18.1 in Schrijver (1997).

3 Linear Relaxation and Strength of IP Formulations

Despite these negative results, a substantial body of theory and very scalable algorithms have been developed to solve IPs. In the subsequent discussion, we focus on optimization problems with rational entries: $A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m, c \in \mathbb{Q}^n$, and we assume that the **feasible set of the IPs of interest is bounded**. Rational entries are needed both for theoretical purposes (e.g., to ensure that optimal solutions exist) but also for the purely practical necessity of representing optimization problems on computers with finite memory. Considering bounded feasible sets only simplifies a few statements, but is not really needed for any of the theory. (In practice, this is not a terrible assumption anyway because we rarely deal with optimization problems that are truly unbounded!)

Let us start in the same way we started with linear optimization, and consider the problem of finding a good **lower bound** for an IP. Clearly, we could obtain a bound if we relaxed the integrality requirements. The following definition allows us to formalize this.

Definition 1 (LP relaxation)**.** *Given the generic integer program:*[2]

$$\begin{aligned} \min\ & c^\intercal x + d^\intercal y \\ & Ax + By = b \\ & x, y \geq 0 \\ & x \in \{0,1\}^{n_1}, y \in \mathbb{Z}^{n_2}, \end{aligned}$$

its linear programming relaxation is obtained by replacing the requirement $x \in \{0,1\}^{n_1}$ *with* $x \in [0,1]^{n_1}$ *and replacing the requirement* $y \in \mathbb{Z}^{n_2}$ *with* $y \in \mathbb{R}^{n_2}$*.*

The **LP relaxation** entails changing the binary requirement on x into a (continuous) restriction to the interval $[0,1]$ and removing the integrality requirement on y. The feasible set of the original IP is therefore contained in the feasible set of its LP relaxation (which also justifies the name!). The following observation is immediate.

Observation 1. *The optimal value of the LP relaxation to an IP provides a lower bound on the optimal value of the IP. Moreover, if the optimal solution to the LP relaxation is feasible for the original IP, then that solution is optimal for the IP.*

In practice, the LP relaxation could be quite strong but also quite weak, and critically, **this depends on the formulation of the IP!** To appreciate this point, let us consider again some of our earlier motivating examples.

3.1 Strength of IP Formulations in Our Examples

3.1.1 Facility Location.

In §1.2, we presented an IP formulation for the facility location problem. For convenience, we replicate it here (omitting the objective) and we also introduce a new formulation for the feasible set that we refer to as the **aggregate facility location (AFL)** formulation:

$$\begin{aligned} &(FL) \\ &\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, m \\ &x_{ij} \leq y_j, \quad i = 1, \ldots, m, \quad j = 1, \ldots, n \\ &x_{ij}, y_j \in \{0,1\} \end{aligned} \qquad \begin{aligned} &(AFL) \\ &\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \ldots, m \\ &\sum_{i=1}^{m} x_{ij} \leq m y_j, \quad j = 1, \ldots, n \\ &x_{ij}, y_j \in \{0,1\}. \end{aligned}$$

The main difference is that (AFL) replaces the constraints $x_{ij} \leq y_j$ in (FL) with the constraints $\sum_{i=1}^{m} x_{ij} \leq m y_j$. Because the latter constraint forces x_{ij} to be 0 whenever $y_j = 0$ but allows x_{ij} to be 1 if $y_j = 1$, it is a valid reformulation. So the two formulations **result in the same feasible set of integer points** x, y and therefore also **the same optimal solutions and optimal costs**.

[2] A similar definition also applies to mixed-integer problems. In that case, restrictions on any continuous variables would remain unchanged.

On first glance, the (AFL) formulation might seem superior because it has $m + n$ constraints, whereas the (FL) formulation has $m + m \cdot n$ constraints.

But consider their corresponding LP relaxations. We define the following two polyhedra, which are the feasible sets of the two relaxations:

$$P_{\text{FL}} = \left\{ (x, y) : \sum_{j=1}^{n} x_{ij} = 1, \forall i, \quad x_{ij} \leq y_j, \forall i, j, \quad 0 \leq x_{ij} \leq 1, \quad 0 \leq y_j \leq 1 \right\}$$

$$P_{\text{AFL}} = \left\{ (x, y) : \sum_{j=1}^{n} x_{ij} = 1, \forall i, \quad \sum_{i=1}^{m} x_{ij} \leq m \cdot y_j, \forall j, \quad 0 \leq x_{ij} \leq 1, \quad 0 \leq y_j \leq 1 \right\}$$

Clearly, $P_{\text{FL}} \subseteq P_{\text{AFL}}$ and the inclusion can actually be **strict**. In other words, the feasible set of the LP relaxation for formulation (FL) is closer to the set of integer solutions than the LP relaxation of formulation (AFL). The situation corresponds visually to Figure 6.

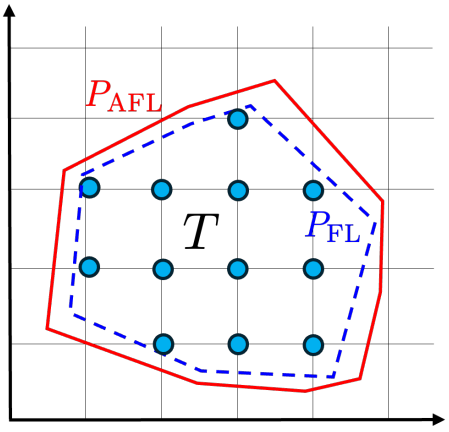


Figure 6: The feasible sets P_{FL} and P_{AFL} for the two LP relaxations for the facility location problem. Note that the feasible points T for the IP (and their convex hull) are the only integer points contained in both P_{FL} and P_{AFL}, but the (FL) formulation provides a tighter relaxation than the (AFL) formulation.

So if Z_{IP} is the optimal cost of the facility location IP and Z_{FL} and Z_{AFL} are the optimal costs of the two LP relaxations, we obtain that:

$$Z_{\text{AFL}} \leq Z_{\text{FL}} \leq Z_{\text{IP}},$$

so **(FL) provides a better (i.e., higher) lower bound on optimal cost than (AFL)**.

3.1.2 Minimum Spanning Tree Revisited

Recall the minimum spanning tree (MST) construction and the two formulations – cutset and subtour-elimination – which we replicate below for convenience.

(Cutset MST)

$$\sum_{e\in\mathcal{E}} x_e = n-1,$$
$$\sum_{e\in\delta(S)} x_e \geq 1, \quad S\subset\mathcal{N}, S\neq\emptyset$$
$$x_e \in \{0,1\}$$

(Subtour-elimination MST)

$$\sum_{e\in\mathcal{E}} x_e = n-1,$$
$$\sum_{e\in\mathcal{E}(S)} x_e \leq |S|-1, \quad S\subset\mathcal{N}, S\neq\emptyset,$$
$$x_e \in \{0,1\}.$$

Theorem 2. *With P_{cut} and P_{sub} denoting the feasible sets of the two LP relaxations,*
i) $P_{sub}\subseteq P_{cut}$ and examples exist where $P_{sub}\subset P_{cut}$.
ii) P_{cut} can have fractional extreme points.

Proof. a) For any set S of nodes, we have

$$\mathcal{E} = \mathcal{E}(S)\cup\delta(S)\cup\mathcal{E}(\mathcal{N}\setminus S).$$

Therefore,

$$\sum_{e\in\mathcal{E}(S)} x_e + \sum_{e\in\delta(S)} x_e + \sum_{e\in\mathcal{E}(\mathcal{N}\setminus S)} x_e = \sum_{e\in\mathcal{E}} x_e.$$

For $x\in P_{\text{sub}}$, and for $S\neq\emptyset, N$, we have

$$\sum_{e\in\mathcal{E}(S)} x_e \leq |S|-1,$$

and

$$\sum_{e\in\mathcal{E}(\mathcal{N}\setminus S)} x_e \leq |N\setminus S|-1.$$

Because

$$\sum_{e\in\mathcal{E}} x_e = n-1,$$

we obtain that

$$\sum_{e\in\delta(S)} x_e \geq 1,$$

and therefore $x\in P_{\text{cut}}$.

b) We refer the interested reader to Bertsimas and Tsitsiklis (1997) for an example. □

3.1.3 Traveling Salesperson Problem Revisited

Lastly, recall the cutset and subtour-elimination formulations for the TSP.

$$\begin{aligned} &\textbf{(Cutset TSP)} && \textbf{(Subtour-elimination TSP)} \\ &\sum_{e\in\delta(\{i\})} x_e = 2, \forall i \in N && \sum_{e\in\delta(\{i\})} x_e = 2, \forall i \in N \\ &\sum_{e\in\delta(S)} x_e \geq 2, \forall S \subset N, S \neq \emptyset && \sum_{e\in\mathcal{E}(S)} x_e \leq |S| - 1, \forall S \subset N, S \neq \emptyset. \end{aligned}$$

Letting P_{TScut} and P_{TSsub} be the polyhedra corresponding to the LP relaxations of these two formulations, it turns out that the two formulations are equally strong, i.e., $P_{\text{TScut}} = P_{\text{TSsub}}$ (see Bertsimas and Weismantel (2005) and Bertsimas and Tsitsiklis (1997) for proofs.)

3.2 Strength of IP Formulation

These examples show that different formulations of the IP could result in different LP relaxations and therefore different lower bounds on the IP's optimal value. Because we no longer have strong duality, the quality of the lower bounds will be **critical** when solving IPs, so it is important to understand what makes some formulations better than others – and also consider what an "ideal" formulation could look like.

To understand this, let T denote all the feasible points to an IP, define

$$\text{conv}\,(T) = \left\{ \sum_{x\in T} \lambda_x \cdot x \;:\; \lambda \geq 0,\, e^\intercal \lambda = 1 \right\}$$

as their convex hull, and let P denote the feasible (polyhedral) region of an LP relaxation to our IP. Because we assumed that the feasible set for the IP is bounded, the set T is finite and $\text{conv}\,(T)$ **is a polyhedral set**! Then, we clearly have (see Figure 7 for a visualization):

$$T \subseteq \text{conv}\,(T) \subseteq P.$$

This shows that the **ideal** LP relaxation would be one that exactly corresponds to $\text{conv}\,(T)$! Put differently, if we had access to an explicit representation of $\text{conv}\,(T)$ – for instance, as an inequality description $\text{conv}\,(T) = \{x : Dx \leq d\}$ – then we could immediately solve our IP by **solving a linear program on the polyhedral set** $\text{conv}\,(T)$.

We highlight some important take-aways, which we summarize in the following remarks.

Remark 1 (Quality of formulations)**.** ***The quality of a formulation*** *for an IP with feasible set T can be judged by how closely its LP relaxation approximates* $\text{conv}\,(T)$*. In particular, for two formulations A and B with the same feasible set of integer points and with P_A and P_B denoting the feasible sets of their LP relaxations, A is said to be* ***stronger*** *(i.e., results in an improved lower bound) than B if $P_A \subset P_B$.*

Remark 2 (Models with more constraints)**.** ***Constraints*** *play a more subtle role in an IP formulation than in an LP formulation. Whereas in an LP, formulations with more*

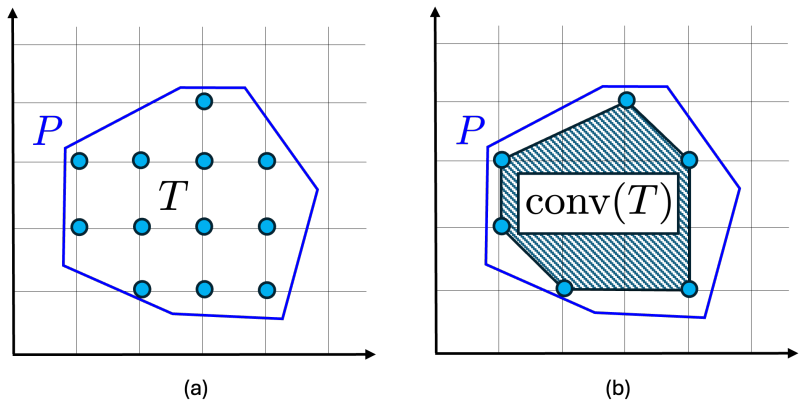


Figure 7: (a) Depicts the feasible set of the LP relaxation – the polyhedron P – and the set T of all the integer points in P. (b) Depicts the convex hull of the integer points, conv (T). The optimal value for the IP is same as the optimal value over the set conv (T).

constraints should be avoided[3]*, a valid IP formulation with more constraints is typically stronger. Adding more (valid) constraints in an IP formulation thus involves a trade-off between the strength and the and the size of the formulation.*

The results in this section will lead us in two different directions. §4 examines what types of IP formulations are "ideal," meaning they result in LP relaxations that exactly correspond to the convex hull of all integer feasible solutions. When that is not possible, §6 shows how to add **valid cuts**, which are linear inequalities that remove fractional points from the feasible set of the LP relaxation without removing any integer points.

4 Ideal Formulations With Total Unimodularity

This section examines the first set of conditions that guarantee an ideal IP formulation, i.e., one where the LP relaxation's feasible region would have only **integral** extreme points.

Let $\mathcal{F} = \{x \in \mathbb{Z}^n_+ \mid Ax \leq b\}$ be the set of integer points for an IP formulation, where $A \in \mathbb{Z}^{m \times n}$ and $b \in \mathbb{Z}^m$,[4] and let P denote the feasible set of its LP relaxation:

$$P = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}.$$

Our goal is to identify conditions on the matrix A such that P **is integral**, i.e., $P = \text{conv}(\mathcal{F})$. We start by recalling Cramer's rule.

[3]For LPs, introducing constraints increases the problem size and also introduces degeneracy, which can complicate algorithms like simplex.

[4]The restriction to integer matrices is without loss of generality here: if the entries were rational, we could multiply all the equations by the least common multiple of (the absolute values of) all denominators.

Proposition 1 (Cramer's Rule)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. For $b \in \mathbb{R}^n$,*

$$Ax = b \quad \Longrightarrow \quad x = A^{-1}b \quad \Longrightarrow \quad x_i = \frac{\det(A^i)}{\det(A)},\ \forall i,$$

where A^i is the matrix with columns $A^i_j = A_j$ for all $j \in \{1,\dots,n\} \setminus \{i\}$ and $A^i_i = b$.

To motivate the definition of total unimodularity, consider the polyhedron

$$P = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$$

with $A \in \mathbb{Z}^{m\times n}$ of full row rank and $b \in \mathbb{Z}^m$. For each vertex x of P, there exists a basis $B \subset \{1,\dots,n\}$ such that $x_B = A_B^{-1}b$ and $x_N = 0$. For matrices with $\det(A_B) = \pm 1$, Cramer's rule ensures that A_B^{-1} is integral. Therefore, integrality of x can be guaranteed if we require that $\det(A_B)$ is equal to ± 1. This motivates the following definition.

Definition 2 (Unimodularity, Total unimodularity)**.**

1. *A matrix $A \in \mathbb{Z}^{m\times n}$ of full row rank is **unimodular** if the determinant of A_B is 1 or -1 for every basis B.*
2. *A matrix $A \in \mathbb{Z}^{m\times n}$ is **totally unimodular** if the determinant of each square submatrix of A is 0, 1, or -1.*

Note that all entries of a totally unimodular matrix (which are 1×1 submatrices of A) must belong to the set $\{0, 1, -1\}$. However, that is not the case for unimodular matrices; for instance, the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

is unimodular. The reason we carry both definitions is to be able to make statements about optimization problems in standard form and in inequality form. We will provide several characterizations that allow checking quickly whether a matrix is (totally) unimodular. For now, to appreciate why the definitions are important, we state the main result of interest.

Theorem 3.

1. *The matrix $A \in \mathbb{Z}^{m\times n}$ of full row rank **is unimodular if and only if** the polyhedron $P(b) = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$ is integral for all $b \in \mathbb{Z}^m$ with $P(b) \neq \emptyset$.*
2. *The matrix A **is totally unimodular if and only if** the polyhedron $P(b) = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ is integral for all $b \in \mathbb{Z}^m$ with $P(b) \neq \emptyset$.*

Proof. (a) "⇒" Assume that A is unimodular. Consider $b \in \mathbb{Z}^m$ with $P(b) \neq \emptyset$. Any extreme point $x \in P(b)$ can be written as (x_B, x_N), where $x_B = A_B^{-1}b$ and $x_N = 0$ for some basis B. Because A is unimodular, $\det(A_B) = \pm 1$, which by Cramer's rule implies that x_B (and therefore also x) is integral.

"$\Leftarrow$" Suppose that $P(b) \neq \emptyset$ is integral for any integral b. Let B be any basis of A. We claim that it's sufficient to argue that A_B^{-1} is integral; because A_B is integral and $\det(A_B) \cdot \det(A_B^{-1}) = 1$, that would imply that $\det(A_B) \in \{1, -1\}$ and thus that A is unimodular. To prove that A_B^{-1} is integral, consider a right-hand-side $b = A_B \cdot z + e_i$, where z is an integral vector. We have that $A_B^{-1} \cdot b = z + A_B^{-1} e_i$. Thus, by choosing z sufficiently large so that $z + A_B^{-1} e_i \geq 0$ (which can readily be done by increasing the entries), we obtain a basic feasible solution for $P(b)$. Because this is integral by assumption, this implies that $A_B^{-1} e_i$ must be integral. Repeating the argument for all e_i proves that A_B^{-1} is integral.

(b) We claim that A is totally unimodular if and only if the matrix $[A, I]$ is unimodular. Moreover, we claim that for any $b \in \mathbb{Z}^m$, the extreme points of the polyhedron $\{x \in \mathbb{R}_+^n \mid Ax \leq b\}$ are integral if and only if the extreme points of the polyhedron $\{(x, y) \in \mathbb{R}_+^{n+m} \mid Ax + Iy = b\}$ are integral. (These will be the subject of future propositions, but the proofs follow by suitably expanding determinants.) The result then follows from part (a). $\square$

The critical consequence from Theorem 3 is that the optimal value in the IP $\min\{c^\intercal x \mid Ax \leq b, x \in \mathbb{Z}_+^n\}$ is obtained by solving the LP $\min\{c^\intercal x \mid Ax \leq b, x \in \mathbb{R}_+^n\}$.

Detecting (total) unimodularity is therefore quite critical, so we provide a few additional characterizations followed by examples.

Proposition 2. *Consider a matrix $A \in \{0, 1, -1\}^{m \times n}$. The following are equivalent:*

1. *A is totally unimodular.*
2. *$A^\intercal$ is totally unimodular.*
3. *$[A^\intercal \ -A^\intercal \ I \ -I]$ is totally unimodular.*
4. *$\{x \in \mathbb{R}_+^n \mid Ax = b, 0 \leq x \leq u\}$ is integral for all integral b, u.*
5. *$\{x \mid a \leq Ax \leq b, \ell \leq x \leq u\}$ is integral for all integral a, b, ℓ, u.*
6. *Each collection of columns of A can be partitioned into two parts so that the sum of the columns in one part minus the sum of the columns in the other part is a vector with entries $0, +1$, and -1. (By part 2, a similar result also holds for the rows of A.)*
7. *Each nonsingular submatrix of A has a row with an odd number of non-zero components.*
8. *The sum of entries in any square submatrix with even row and column sums is divisible by four.*
9. *No square submatrix of A has determinant +2 or -2.*

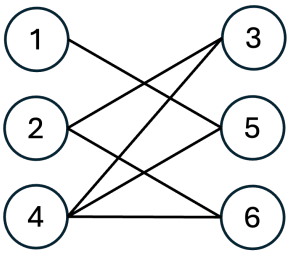
For a proof of these results, see Theorem 19.3 in Schrijver (1997).

4.1 Examples of Totally Unimodular Matrices

4.1.1 Node-Edge Incidence Matrix for Bipartite Graphs

Let $G = (\mathcal{N}, \mathcal{E})$ be an **undirected graph** and let $A \in \{0,1\}^{|\mathcal{N}| \times |\mathcal{E}|}$ be the node-edge incidence matrix of G, i.e., $A_{i,e} = 1$ if and only if $i \in e$. Then, A **is TU if and only if the graph** G **is bipartite**. The proof follows from Proposition 2 (#6) and is omitted.

Figure 8 shows an example. Recalling our discussion of matching problems in §1.6, it can be seen that if these are defined on bipartite graphs, all matching formulations will admit integral LP relaxations.

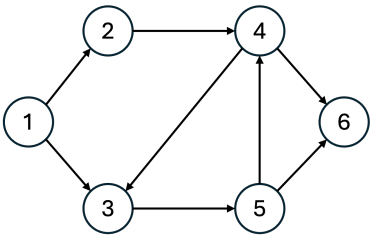


	$\{1,5\}$	$\{2,3\}$	$\{2,6\}$	$\{4,3\}$	$\{4,5\}$	$\{4,6\}$
1	1	0	0	0	0	0
2	0	1	1	0	0	0
3	0	1	0	1	0	0
4	0	0	0	1	1	1
5	1	0	0	0	1	0
6	0	0	1	0	0	1

Figure 8: Undirected bipartite graph and its node-edge incidence matrix.

4.2 Node-Arc Incidence Matrix for Directed Graphs

Let $D = (N, E)$ be a directed graph and let M be the $N \times E$ incidence matrix of D, where $M_{v,e} = 1$ if and only if $e = (\cdot, v)$ (arc e enters node v), $M_{v,e} = -1$ if and only if $e = (v, \cdot)$ (arc e leaves node v), and $M_{v,e} = 0$ otherwise. Then, M is TU. Figure 9 shows an example.



	$(1,2)$	$(1,3)$	$(2,4)$	$(4,3)$	$(3,5)$	$(5,4)$	$(4,6)$	$(5,6)$
1	-1	-1	0	0	0	0	0	0
2	1	0	-1	0	0	0	0	0
3	0	1	0	1	-1	0	0	0
4	0	0	1	-1	0	1	-1	0
5	0	0	0	0	1	-1	0	-1
6	0	0	0	0	0	0	1	1

Figure 9: Directed graph and its node-arc incidence matrix.

The important consequence of this result is that all network flow problems with integral arc capacities and integral demand or supply at nodes will admit an integral LP relaxation. The Prosche Motors problem on the second homework is one such example – so with integral data, the optimal solution is guaranteed to be integral.

4.3 Interval Matrices

If $A \in \{0,1\}^{m \times n}$ and each column of A has its values of 1 consecutively (under some ordering of the columns of A), then A is TU. Such matrices are called **interval matrices**.

An example is the matrix below.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

4.4 Network Matrices

All of the examples above are special instances of **network matrices**. To formalize this broad class, let $D = (V, A)$ be a directed graph and let $T = (V, A_0)$ be a directed tree on V. Let M be the $A_0 \times A$ matrix defined by, for $a = (v, w) \in A$ and $a' \in A_0$:

$$M_{a',a} = \begin{cases} +1 & \text{if the unique } v - w \text{ path in } T \text{ passes through } a' \text{ forwardly} \\ -1 & \text{if the unique } v - w \text{ path in } T \text{ passes through } a' \text{ backwardly} \\ 0 & \text{if the unique } v - w \text{ path in } T \text{ does not pass through } a'. \end{cases}$$

Then, M is TU. For an example, consider the directed graph D and directed tree T in Figure 10. The corresponding network matrix is shown in Table 1.

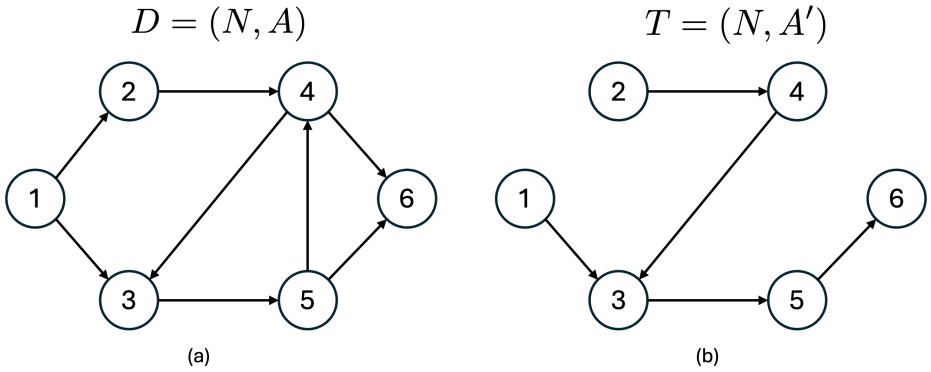


Figure 10: Directed graph (a) and a directed tree (b).

	$(1,2)$	$(1,3)$	$(2,4)$	$(4,3)$	$(3,5)$	$(5,4)$	$(4,6)$	$(5,6)$
$(1,3)$	1	1	1	0	0	0	0	0
$(2,4)$	−1	0	0	0	0	0	0	0
$(4,3)$	−1	0	0	1	0	−1	1	0
$(3,5)$	0	0	0	0	1	−1	1	0
$(5,6)$	0	0	0	0	0	0	1	1

Table 1: Network matrix corresponding to the directed graph and tree in Figure 10.

There is a very famous result in combinatorial optimization due to Seymour (1980), who showed that every TU matrix arises, in a certain way, from network matrices and just **two** other matrices. Importantly, testing whether a matrix is TU can be done in polynomial time; for more details, see Schrijver (1997).

5 Dual Integrality and Submodular Functions

Next, we discuss an alternative way to show integrality of polyhedra based on linear optimization duality. This will also allow us to discuss submodular and supermodular functions, which are extremely important concepts in their own right in optimization.

The approach here is based on a simple observation: to show that the LP relaxation of an IP has integral extreme points, it suffices to check that the optimal value of any LP with integer cost vector is an integer. The following proposition summarizes the idea.

Proposition 3. *Let P be a nonempty polyhedron with at least one extreme point. The polyhedron P is integral if and only if the optimal value $Z_{LP} := \min\{c^\intercal x \mid x \in P\}$ is an integer, for all $c \in \mathbb{Z}^n$.*

The intuition should be quite clear; the proof is straightforward and is omitted.

Therefore, to show integrality of a polyhedron P, it suffices to show that $Z_{LP} \in \mathbb{Z}$ for all $c \in \mathbb{Z}^n$. One way to show that is to **construct a dual optimal integral solution** for any such c.[5] We illustrate this with an example that is important in its own right.

5.1 Polymatroid Polyhedra and Submodular Functions

For a given finite set $N = \{1, \ldots, n\}$, consider a function $f(S)$ defined on subsets S of N.

Definition 3 (Sub-, super-modular)**.** *A set function $f : 2^N \to \mathbb{R}$ is **submodular** if*

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T), \quad \forall S, T \subset N$$

*and it is **supermodular** if the reverse inequality holds.*

Note that the condition in the definition may not make a lot of sense written this way, but it is equivalent to:

$$f(S) - f(S \cap T) \geq f(S \cup T) - f(T), \quad \forall S, T \subset N.$$

In this form, note that the set difference between the sets appearing on the left of the inequality is exactly $S \setminus (S \cap T) = S \setminus T$, which exactly matches the difference between the sets on the right because $(S \cup T) \setminus T = S \setminus T$. So the condition is stating that the gains obtained when adding $S \setminus T$ to the set $S \cap T$ are greater than the gains obtained when adding the same set to the larger set T. The following alternative definitions will make this intuition even more clear.

[5] These ideas are related to the concept of **total dual integrality** (TDI), which has been studied extensively in combinatorial optimiziation. For a more general treatment, see Schrijver (1997).

Proposition 4. *A set function $f : 2^N \to \mathbb{R}$ is **submodular if and only if**:*

(a) For any $S, T \subseteq N$ such that $S \subseteq T$ and $k \notin T$:

$$f(S \cup \{k\}) - f(S) \geq f(T \cup \{k\}) - f(T).$$

(b) For any $S \subseteq N$ and any j, k with $j, k \notin S$ and $j \neq k$:

$$f(S \cup \{j\}) - f(S) \geq f(S \cup \{j, k\}) - f(S \cup \{k\}). \tag{3.2}$$

For a proof, see Bertsimas and Weismantel (2005) or Bach (2010).

These equivalent definitions should make it clear that a submodular function has the certain **"diminishing returns"** or **"decreasing differences"** property: the marginal gain when adding an element k to a larger set T is smaller than the gain when adding k to a smaller set S (or equivalently, the marginal gain from including an extra element j is smaller when some other element k is also included). In economics, a submodular **cost** function captures economies of scale, whereas a submodular **profit function** captures substitution. (Supermodular functions are the exact opposite.) On first glance, one may perceive submodular functions as a discrete analog to concave functions, but that analogy only holds solely in terms of economic intuition, but **not** from an optimization standpoint! In fact, **in terms of optimization, submodular functions behave more like convex functions**, e.g., there are efficient algorithms to minimize them, they admit a very elegant link to convexity (through the Lovasz extension) and they also admit a duality theory.

Submodular and supermodular functions play central roles in a variety of fields, including operations research, economics, and computer science. The scope of our treatment here will be limited, but we direct the interested reader to Bach (2013) for a concise overview and the book Schrijver (2003) for an in-depth treatment.

Subsequently, we are interested in submodular functions that are non-negative and **increasing**[6] in the set inclusion sense, i.e.,

$$f(S) \leq f(T), \quad \forall S \subset T \subseteq N.$$

5.1.1 Examples

A few quick examples of (monotone) submodular functions.

- **Linear functions.** A function $f : 2^N \to \mathbb{R}$ is **modular** if

$$f(A) = \sum_{i \in A} w_i$$

for some weights $w : N \to \mathbb{R}$. Such functions are both supermodular and submodular. If $w_i \geq 0$ for all $i \in N$, then f is also increasing.

[6] We use "increasing" and "decreasing" in weak sense, and use "strictly" to emphasize strict relationships.

- **Compositions with linear functions.** As a generalization of the linear case, consider any weights $w \geq 0$ and any **concave** function $g : \mathbb{R} \to \mathbb{R}$. Then, the function $f : 2^N \to \mathbb{R}$

$$f(S) = g\Big(\sum_{i \in S} w_i\Big)$$

is submodular. If g is increasing, f is also increasing.

For a different example, suppose that g is **convex** and the weights w are zero except for two weights with opposite signs, i.e., $\exists\, i \neq j : w_i \leq 0,\, w_j \geq 0$. Then, $f : 2^N \to \mathbb{R}$ defined as

$$f(S) = g\Big(\sum_{i \in S} w_i\Big)$$

is submodular.

- **Set systems and coverage functions.** Given a universe U and n subsets $A_1, A_2, \ldots, A_n \subset U$, we can define several natural submodular functions on the set $N = \{1, 2, \ldots, n\}$. First, the coverage function given by

$$f(S) = \left| \bigcup_{i \in S} A_i \right|$$

is submodular. This naturally extends to the weighted coverage function: given a non-negative weight function $w : U \to \mathbb{R}_+$,

$$f(S) = w\left(\bigcup_{i \in S} A_i \right)$$

is submodular. Another related function defined by

$$f(S) = \sum_{x \in U} \max_{i \in S} w(A_i, x)$$

is also submodular, where $w(A_i, x)$ is a non-negative weight for A_i covering x. All these functions are increasing.

- **Valuation functions with decreasing marginal values**. Sometimes we assume that a certain function is submodular not because it arises in a specific combinatorial way, but because it arises in a setting where it's natural to have decreasing marginal returns. An example are combinatorial auctions, where each player has a valuation function $w : 2^N \to \mathbb{R}$ on subsets of items. This might have a specific form, like

$$w(S) = \min\left(\sum_{j \in S} v_j, B \right),$$

or it might be given by a black box. However, we might assume that the (unknown) function is submodular just it may be natural to expect that having more items decreases the benefit of acquiring another item.

- **Optimal TSP cost on tree graphs.** Consider an **undirected tree graph** $G = (N, E)$ with a positive cost for traversing the edges ($c_e \geq 0$ for every edge $e \in E$). For every $S \subseteq N$, define $f(S)$ as the optimal (i.e., smallest) cost for a TSP that goes through all the nodes in S. Then, $f(S)$ is submodular.
- **Network optimization.** Submodular functions also arise in network optimization models. For instance, consider a directed graph where there are capacities on the edges that constrain how much flow can be transported through the edge. Then, if we define $f(S)$ as the maximum flow that can be received at a set of sink nodes S, the function $f(S)$ is submodular.
- **Inventory and supply chain management.** Lastly, submodular functions appear frequently in the study of supply chain and inventory management, such as when characterizing perishable inventory systems, dual sourcing, and inventory control problems with trans-shipment.

Returning to our setting, let us consider the following problem:

$$\begin{aligned} \text{maximize } & \sum_{j=1}^{n} r_j x_j \\ & \sum_{j \in S} x_j \leq f(S), \ \forall S \subseteq N \\ & x \in \mathbb{Z}_+^n. \end{aligned}$$

This problem essentially looks like an extension of the knapsack problem that we considered earlier, except that there is one constraint for every possible subset $S \subseteq N$. Let $\mathcal{F}$ denote the set of feasible integer solutions and let

$$P(f) = \left\{ x \in \mathbb{R}_+^n \;\middle|\; \sum_{j \in S} x_j \leq f(S), \ \forall S \subset N \right\}$$

denote the feasible set of the LP relaxation.

We next state and prove the main result in this section: the polyhedron $P(f)$, which is called a **polymatroid**, is integral for any $f(S)$ is submodular and increasing.

Theorem 4. *If f is submodular, increasing, integer valued, and $f(\emptyset) = 0$, then*

$$P(f) = \text{conv}(\mathcal{F}).$$

Proof. Consider the linear relaxation and its dual:

$$\begin{aligned} \text{maximize } & \sum_{j=1}^{n} r_j x_j \\ & \sum_{j \in S} x_j \leq f(S), \quad S \subset N, \\ & x_j \geq 0, \ j \in N \end{aligned} \qquad \begin{aligned} \text{minimize } & \sum_{S \subset N} f(S) y_S \\ & \sum_{S : j \in S} y_S \geq r_j, \ j \in N, \\ & y_S \geq 0, \quad S \subset N. \end{aligned} \tag{7}$$

The key intuition behind the proof is that in a maximization like the one in the primal above, the use of a submodular function to evaluate the right-hand-sides implies that a **greedy** heuristic actually produces an optimal solution. So we will construct such a greedy solution for the primal and also a feasible solution for the dual with the same cost.

Suppose $r_1 \geq r_2 \geq \ldots \geq r_k > 0 \geq r_{k+1} \geq \ldots \geq r_n$. Let $S^j = \{1, \ldots, j\}$ for $j \in N$, and $S^0 = \emptyset$. We prove that the following primal and dual solutions x and y are optimal for the primal and dual problem, respectively.

$$x_j = \begin{cases} f(S^j) - f(S^{j-1}), & \text{for } 1 \leq j \leq k, \\ 0, & \text{for } j > k. \end{cases}$$

$$y_S = \begin{cases} r_j - r_{j+1}, & \text{for } S = S^j, \quad 1 \leq j < k, \\ r_k, & \text{for } S = S^k, \\ 0, & \text{otherwise.} \end{cases}$$

Because f is integer valued, $x \in \mathbb{Z}^n$. Moreover, x is primal feasible: f is increasing, which implies $x_j \geq 0$, and for all $T \subset N$, we have:

$$\begin{aligned} \sum_{j \in T} x_j &= \sum_{j \in T, j \leq k} \left(f(S^j) - f(S^{j-1}) \right) \\ \text{(because } f \text{ submodular)} \quad &\leq \sum_{j \in T, j \leq k} \left(f(S^j \cap T) - f(S^{j-1} \cap T) \right) = \\ &= f(S^k \cap T) - f(\emptyset) \\ \text{(because } f \text{ monotone)} \quad &\leq f(T) - f(\emptyset) \\ \text{(because } f(\emptyset) = 0) \quad &= f(T). \end{aligned}$$

To show that y is dual feasible, note that $y_S \geq 0$ and:

$$\sum_{S: j \in S} y_S = y_{S^j} + \ldots + y_{S^k} = r_j, \text{ if } j \leq k \quad \text{and} \quad \sum_{S: j \in S} y_S = 0 \geq r_j, \text{ if } j > k.$$

The primal objective value is

$$\sum_{j=1}^{k} r_j \left(f(S^j) - f(S^{j-1}) \right),$$

and the dual objective value is

$$\sum_{j=1}^{k-1} (r_j - r_{j+1}) f(S^j) + r_k f(S^k) = \sum_{j=1}^{k} r_j \left(f(S^j) - f(S^{j-1}) \right).$$

From strong duality, the two problems have the same optimal value. Because this is true for every $r \in \mathbb{Z}^n$, it follows that $P(f) = \text{conv}\,(\mathcal{F})$. □

An analogous result holds in the context of the following minimization problem

$$\begin{aligned} \min \ & \sum_{j=1}^{n} c_j x_j \\ & \sum_{j \in S} x_j \geq f(S), \ \forall S \subseteq N, \\ & x \in \mathbb{Z}_+^n, \end{aligned}$$

where the function f is **supermodular**. The arguments are identical and are omitted.

Importantly, the proof above highlighted that a **greedy solution is optimal** for problem (7). The intuition is directly tied to the diminishing returns property of submodular functions and can appreciated when interpreting the problem as a generalized knapsack problem. Because any item j brings more reward when included in a smaller (rather than larger) set S, it is optimal to include the items in decreasing order of their rewards r_i as long as the rewards are positive.

Several important extensions of this result are possible. For instance, a similar result also holds when $f(S) = \min(f_1(S), f_2(S))$ where f_1, f_2 are both submodular, increasing, integer-valued functions. For details, see Bertsimas and Weismantel (2005) or Schrijver (2003).

6 Improving LP Relaxations with Valid Cuts

We argued earlier that if T is the set of feasible integer solutions and P is the feasible region of the linear relaxation for an IP formulation, then unless $\text{conv}\,(T) = P$, there will be valid inequalities for $\text{conv}\,(T)$ that are not valid for P. Such inequalities are called **cuts** because they cut off some fractional solution from P. Adding cuts strengthens the IP formulation and also leads to algorithmic improvements.

To appreciate how cuts can be generated, consider the following IP:

$$\begin{aligned} \min \ & c^\intercal x \\ & Ax = b \\ & x \geq 0 \\ & x \in \mathbb{Z}^n, \end{aligned}$$

where A, b, c have rational entries. The LP relaxation would be an LP in standard form, so let x^* be an optimal basic feasible solution and B be the associated optimal basis. Then, we can write $x^* = [x_B^*; x_N^*]$ where $x_N^* = 0$ are the nonbasic variables. (For simplicity, we assume here that $B = \{1, \ldots, m\}$, to avoid extra notation.) Then, recall that we have:

$$x_B^* + A_B^{-1} A_N x_N^* = A_B^{-1} b,$$

and consider one of these equalities in which the right-hand-side is **fractional**. Suppose this corresponds to the basic variable x_i, so we write this as:

$$x_i^* + \sum_{j \in N} \bar{a}_{ij} x_j^* = \bar{b}.$$

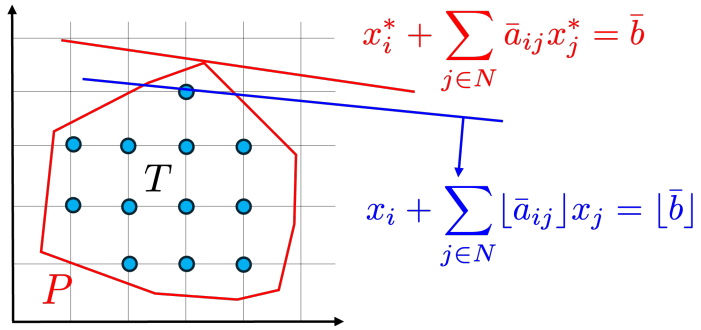


Figure 11: T denotes the feasible set for an IP, and P is the feasible set of the LP relaxation. The red binding constraint $x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}$ defines a supporting hyperplane for P, and the (blue) inequality $x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq x_i + \sum_{j \in N} a_{ij} x_j \leq \lfloor \bar{b} \rfloor$ is a Gomory cut.

But then, because any $x \in T$ must satisfy $x \geq 0$, it can be shown that:

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq x_i + \sum_{j \in N} a_{ij} x_j = \bar{b}, \, \forall \, x \in T.$$

Moreover, because $x \in T$ should be integer, it must be that:

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq x_i + \sum_{j \in N} a_{ij} x_j \leq \lfloor \bar{b} \rfloor, \, \forall \, x \in T.$$

This inequality is **satisfied by all integer solutions**, but is **not satisfied by** x^*. (That follows because $x_i^* = \bar{b}$ and $x_j^* = 0$ for all nonbasic $j \in N$, and because $\lfloor \bar{b} \rfloor < \bar{b}$ by our assumption that $\bar{b}$ is fractional.) So we just obtained a valid inequality that cuts off some of the polyhedral region of the feasible relation without removing any feasible **integer** point! This is the key idea behind Gomory cuts (named after Ralph Gomory, who introduced them in 1958). See Figure 11 for a visualization.

This approach lead to the first **cutting plane method** for solving IPs. It has been shown that by systematically adding these Gomory cuts, and using the dual simplex method[7] with appropriate anticycling rules, we obtain a finitely terminating algorithm for solving general IPs. In practice, however, this method has not been particularly successful.

6.1 Inequality Form and Chvatal Closure

If you are wondering how this would work for linear IPs in inequality form, let $A_1, \ldots, A_n$ be the columns of a rational matrix $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$, and define

$$P = \Big\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} A_j x_j \leq b, \;\; x \geq 0 \} \text{ and } T = \{ x \in P : x \in \mathbb{Z}^n \}.$$

[7] We use the dual simplex method here because we are introducing more constraints in the primal, so the dual readily gives a feasible solution.

Note that if for any feasible $x \in P$ and any vector $u \in \mathbb{R}^m_+$, we readily have:

$$u^\intercal Ax \leq u^\intercal b, \tag{8}$$

because this simply entails multiplying the i-th inequality by u_i and adding all the inequalities up. So any constraint of the form (8) must hold for any point in the LP relaxation's feasible set, $x \in P$. But then, because $x \geq 0$, the following inequality must be valid for feasible integer points, $x \in T$:

$$\sum_{j=1}^{n} \big(\lfloor u^\intercal A_j \rfloor\big) x_j \leq u^\intercal b,$$

and because $x \in \mathbb{Z}^n$, we can strengthen this inequality to

$$\sum_{j=1}^{n} \big(\lfloor u^\intercal A_j \rfloor\big) x_j \leq \lfloor u^\intercal b \rfloor. \tag{9}$$

In summary, inequality (9) is valid for all the feasible integer points $x \in P$ and may cut off some fractional points.

By adding inequalities like (9) to the original inequalities $Ax \leq b$, we can strengthen the LP relaxation. By varying the vector $u \geq 0$, we obtain what is known as the first Chvátal closure of the polyhedron P:

$$P_1 := \left\{ x \in \mathbb{R}^n_+ : Ax \leq b,\ \sum_{j=1}^{n} \big(\lfloor u^\intercal A_j \rfloor\big) x_j \leq \lfloor u^\intercal b \rfloor,\ \forall\, u \geq 0 \right\}. \tag{10}$$

It can be shown that the set P_1 is actually a polyehdral set. (This requires a proof because we may be adding an infinite number of inequalities with the process above! However, a finite number of inequalities suffices – see Theorem 9.4 in Bertsimas and Weismantel (2005) or Theorem 23.1 in Schrijver (1997).)

This process can actually be repeated, i.e., we can apply the procedure outlined above to the polyhedron P_1, etc. A remarkable fact is that after a finite number of such iterations, we are guaranteed to recover exactly the convex hull conv $(()\, T)$, so the process terminates in a finite number of steps. We will not prove this here, but the interested reader can refer to Schrijver (1997) for details and a full proof.

7 Lift-and-Project

The key idea behind the **lift-and-project** approach is to construct polyhedral sets that lie between P and conv (T) as projections of higher-dimensional sets that have a polynomial description. The polyhedron $P \subseteq \mathbb{R}^n$ is first lifted into a higher-dimensional space $\mathbb{R}^{n+p}$, where the formulation is strengthened, and is then projected back onto the original space $\mathbb{R}^n$ to obtain a tigher approximation of conv (T).

The construction is actually remarkably intuitive and it is enlightening to see it one time. Consider a polyhedron $P := \{x \in \mathbb{R}^{n+p}_+ : Ax \geq b\}$ and the mixed-$0,1$ feasible set

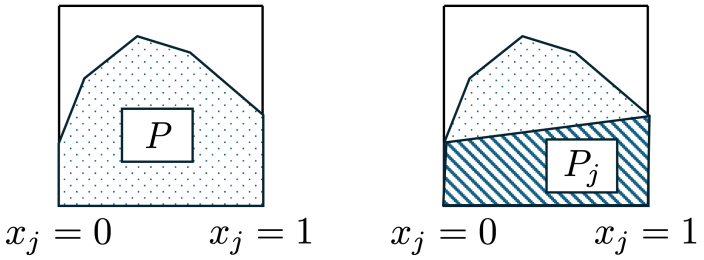


Figure 12: The original polyhedral set P is on the left, and the relaxation P_j obtained via one step of the lift-and-project procedure is on the right.

$T := \{x \in \{0,1\}^n \times \mathbb{R}^p_+ : Ax \geq b\}$ (note that we're generalizing things slightly here by allowing some continuous variables as well). Without loss of generality, we assume the constraints $Ax \geq b$ include $x_j \geq 0$ for $j = 1, \ldots, n+p$, and $x_j \leq 1$ for $j = 1, \ldots, n$. Balas, Ceria, and Cornuéjols study the following **lift-and-project** procedure:

- **Step 0:** Select $j \in \{1, \ldots, n\}$.
- **Step 1:** Generate the nonlinear system:

$$x_j(Ax - b) \geq 0, \quad (1 - x_j)(Ax - b) \geq 0. \tag{11}$$

- **Step 2:** Linearize the system by substituting y_i for $x_i x_j$ (for $i \neq j$), and x_j for x_j^2. (Because $x_j \in \{0,1\}$, note that we have $x_j^2 = x_j$, so the latter substitution is without any loss.) Call the resulting polyhedron M_j.
- **Step 3:** Project M_j onto the x-space. Let P_j be the resulting polyhedron.

We claim that this is a valid relaxation that is better than P, i.e., $T \subseteq P_1 \subseteq P$. That $T \subset P_j$ holds follows because for any $x \in T$, we have $(x, y) \in M_j$ by choosing $y_i = x_i x_j$ for $i \neq j$, because $x_j^2 = x_j$ holds due to x_j being a binary variable.) Moreover, this is a tighter relaxation than P because $Ax \geq b$ is obtained by adding the constraints (11) defining M_j.

The key question is how tight is the relaxation P_j. The following theorem shows that it is actually **the tightest possible among the relaxations that ignore the integrality of all variables x_i for $i \neq j$**. A visualization of the statement is in Figure 12.

Theorem 5. $P_j = \text{conv}\,(\{Ax \geq b,\, x_j = 0\} \cup \{Ax \geq b,\, x_j = 1\})$.

The set $\bigcap_{j=1}^n P_j$ is called the **lift-and-project closure**. It is a better approximation of conv (T) than P,

$$\text{conv}\,(T) \subseteq \bigcap_{j=1}^{n} P_j \subseteq P,$$

and in practice it can be a much better approximation. Bonami and Minoux performed computational experiments on 35 mixed 0-1 linear programs from the MIPLIB library and found that the lift-and-project closure reduces the integrality gap by 37% on average.

Sherali and Adams defined a stronger relaxation by skipping **Step 0** and considering the nonlinear constraints $x_j(Ax - b) \geq 0$ and $(1 - x_j)(Ax - b) \geq 0$ for all $j = 1, \ldots, n$ in Step 1. Then, in Step 2, variables y_{ij} are introduced for all $i = 1, \ldots, n+p$ and $j = 1, \ldots, n$ with $i > j$. Note that the size of the linear system generated in Step 2 is much larger than in the previous lift-and-project procedure: on the order of $n^2 + np$ variables and nm constraints instead of just $n + p - 1$ new variables and $2m$ constraints before. Clearly, the Sherali-Adams relaxation is at least as strong as the lift-and-project closure defined above, and it can be strictly stronger because the Sherali-Adams procedure takes advantage of the fact that $y_{ij} = y_{ji}$, whereas this is not the case for the lift-and-project closure $\bigcap_{j=1}^{n} P_j$.

In practice, experiments done by Bonami and Minoux for the 19 MIPLIB instances for which the Sherali-Adams bound could be computed within an hour show that the improvement over the lift-and-project bound was 10% on average. On these 19 instances, the Sherali-Adams relaxation closed 39% of the integrality gap on average (compared to 29% for the lift-and-project closure).

The Lovasz-Schrijver relaxation is even stronger than Sherali-Adams, but requires solving semidefinite programming problems (which we have not yet discussed :-)).

7.1 Other Types of Cuts

Apart from the Gomory-Chvátal cuts and the Lift-and-Project procedures that we discussed, many other types of cuts also exist. For instance:

- Mixed-Integer Rounding (MIR) Cuts: These are designed to handle general integer variables by transforming fractional solutions into valid integer bounds.
- **Knapsack Cover Cuts**: Applied to knapsack problems

$$w \geq 0, w^\intercal x \leq K \;\Rightarrow < 2- > \sum_i x_i \leq |C| - 1 \text{ for any } C \;:\; \sum_{i \in C} w_i > K \quad \text{(Cover)}$$

- Clique Cuts: These are used to strengthen a constraint like $\sum_{i=1}^{n} x_i \leq 1$ when some of the binary variables appearing form a **clique**. In that case, for any pair of variables in the clique, we can add the constraint $x_i + x_j \leq 1$.
- Flow Cover and Flow Path Cuts: Specialized cuts for models involving flow variables, such as network flow problems.

Solvers such as Gurobi have several cuts embedded and the solution methods actually involve **branch and cut** methods that combine branching with adding cuts. However, developing good cuts often requires good knowledge of the specific combinatorial structure of the problem and is part art, part science.

We refer readers interested in more details to the excellent tutorial Cornuéjols (2008) and several of the classic textbooks.

8 Solving IPs

Unlike linear programming problems, integer programming problems are very difficult to solve. This should not be surprising in view of the complexity of just finding a feasible

solution, which we discussed in §2. However, given their practical importance, many solution approaches have been devised. There are three main categories of algorithms:

1. **Exact algorithms** that are guaranteed to find an optimal solution, but may take an exponential number of iterations. The most famous among these include: cutting planes, branch and bound, branch and cut, lift-and-project methods, and dynamic programming methods.

2. **Approximation algorithms** that provide a suboptimal solution with a bound on the degree of its suboptimality in polynomial time.

3. **Heuristic algorithms** that provide a suboptimal solution, but typically without any guarantees on its quality. Although the running time is not guaranteed to be polynomial, empirical evidence suggests that some of these algorithms find a good solution fast. Examples include local search methods and simulated annealing.

Moreover, a Lagrangean duality theory can also be developed for integer programming problems to derive lower bounds on the optimal cost. Such bounds are very useful in exact algorithms, as they can allow us to avoid enumerating too many feasible solutions and thus speed up the performance.

We briefly overview some of these developments in this section, but refer the interested reader to classical textbooks such as Schrijver (2003) or Bertsimas and Weismantel (2005) for a more thorough overview.

8.1 Cutting Planes

Our previous discussion of cuts leads to the first class of algorithms for solving IPs. This is called a **cutting plane** method and works roughly as follows:

Generic cutting plane algorithm.

1. Solve the linear relaxation and get an optimal solution x^*

2. If x^* is integer stop

3. If not, add a cut (i.e., linear inequality that all integer solutions satisfy but that x^* does not satisfy) and go to step 1 again.

The exact algorithm obviously depends on the choice of cuts. Pure cutting plane algorithms have not been extremely successful in practice, but adding cuts within a branch-and-bound algorithm (which we discuss next) can significantly speed up the framework so cuts are very meaningful in practice.

8.2 Branch and Bound

Branch and bound uses a "divide and conquer" approach to explore the set of feasible integer points, and leverages the LP relaxations to derive bounds on the optimal cost so as to avoid exploring certain parts of the set of feasible integer solutions.

To understand the key intuition, consider a very simple problem of solving an IP with three binary decisions x, y, z:

$$\begin{aligned} &\min\ c_1 \cdot x + c_2 \cdot y + c_3 \cdot z \\ &A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq b \\ &x, y, z \in \{0, 1\} \end{aligned} \tag{12}$$

where A, b and c_1, c_2, c_3 are rational. Figure 13 depicts one possible implementation of a branch-and-bound algorithm. The steps are roughly as follows:

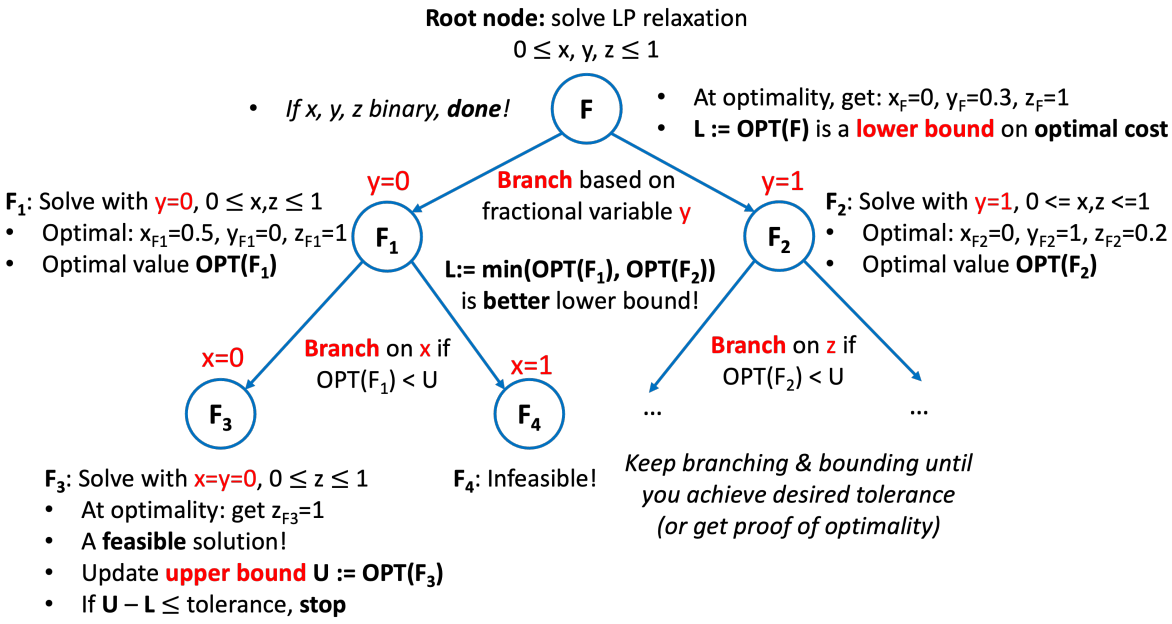


Figure 13: A tree of subproblems in a branch-and-bound procedure.

1. We start with an upper bound $U := +\infty$ and a lower bound $L := -\infty$ on the optimal value of problem (12).

2. In the first step, corresponding to the root node F of the tree, we solve the LP relaxation of (12). If the optimal solution is integral, we are done! This must be optimal in problem (12)! Suppose the optimal value is $OPT(F)$ and the optimal solution involves fractional variables: $x_F = 0, y_F = 0.3, z_F = 1$.

3. Note that now $L := OPT(F)$ is an improved (finite!) lower bound on the problem.

4. We then select one of the variables that was fractional in the optimal solution (here, y) and create two **branches** that correspond to two sub-problems named F_1 and F_2. This step is what gives the method its name – branching. In problem F_1, we solve an

LP relaxation where we constrain y to be 0, whereas in F_2 we solve an LP relaxation where we constrain y to be 1. In both of these problems, the other variables x, z are only constrained to satisfy $0 \leq x, z \leq 1$.

5. Suppose we first solve F_1 and at optimality we get $OPT(F_1)$ and the optimal solution $x_{F1} = 0.5, y_{F1} = 0, z_{F1} = 1$. We do not yet have a feasible solution, so the upper bound is still $U = +\infty$. But we can further branch based on the fractional variable x, so we create two other subproblems, F_3 and F_4. In both of these LPs, y is still constrained to satisfy $y = 0$ (these are part of the same master branch corresponding to F_1), but in F_3 we also add the constraint $x = 0$, whereas in F_2 we add the constraint $x = 1$. In both F_3 and F_4, z is only required to satisfy $0 \leq z \leq 1$.

6. Continuing in this fashion, upon any subsequent branching, we increase the number of decisions that are constrained to belong to $\{0, 1\}$, so eventually we may obtain a feasible solution for problem (12). In our case, that happens in node F_3. (In case this does not happen in the current subtree or in any other subtree, the problem must be infeasible...) Once we have a feasible solution, we can **update the upper bound**. In our case, $U := OPT(F_3)$ is a valid upper bound. If $U - L$ satisfies our desired tolerance/optimality gap, we can even stop the entire process. Otherwise, we can continue this process of branching and bounding.

7. A few additional observations that can speed up the procedure:

   - once we have a finite upper bound U, we need need to explore subtrees for a root note that has optimal value larger than U. For instance, if the LP in node F_2 returns optimal value $OPT(F_2) > U$, that subtree is not worth exploring because the best possible binary solution in that entire subtree is worse than our current best feasible solution!
   - If we end up solving LP relaxations for all the children of a node, the maximum of those optimal values is a better lower bound of the optimal value achievable by IP solutions in that node. For instance, if we solve both F_1 and F_2, then $\min(OPT(F_1), OPT(F_2))$ is a better lower bound for the optimal value of the original problem. This allows updating the lower bound L on the overall problem.

Our example showcases the fundamental principles behind a branch-and-bound approach. For a slightly more general description, let F be the set of feasible solutions to a minimization IP:

$$\text{minimize} \quad c^\intercal x \quad \text{subject to} \quad x \in \{x : Ax \leq b,\, x \in \mathbb{Z}^n\} \tag{13}$$

The fundamental idea is to partition the set F into a finite collection of subsets F_i and solve a separate subproblem for each subset. For every such subproblem, we only need to be able to derive a **lower bound** $\ell(F_i)$ on the cost of the subproblem, i.e., we need to compute

$$\ell(F_i) \leq \min_{x \in F_i} c^\intercal x.$$

At each step, we thus maintain a collection of subproblems/nodes to potentially explore further, and choose one of these to explore. Each subproblem may be almost as difficult

as the original problem and this suggests trying to solve each subproblem by means of the same method, that is, by splitting it into further subproblems. This is the **branching** part of the method.

In the course of this process, we occasionally solve certain subproblems to optimality or simply evaluate the cost of certain feasible solutions. This allows us to maintain an upper bound U on the optimal cost, which could be the cost of the best feasible solution encountered thus far. If the lower bound $\ell(F_i)$ corresponding to a particular subproblem satisfies $\ell(F_i) \geq U$, then this subproblem need not be considered further, since the optimal solution to the subproblem is no better than the best feasible solution encountered thus far.

Importantly, there are several choices to be set in this algorithm; for instance:

1. There are different ways of choosing which subproblem to explore next. Two extreme choices are "breadth-first search" and "depth-first search."

2. There may be several ways of obtaining a lower bound $\ell(F_i)$ on the optimal cost of a subproblem. One possibility that we used above is to consider the LP relaxation. However, we can also employ smart Lagrangean duality to derive **tighter** bounds – see 8.3 for details.

3. The LP relaxations could be improved by **adding cuts**. This is what **branch-and-cut** approaches do.

4. There are several ways to break a problem into subproblems. Our example partitioned based on constraints such as $x_i \leq \lfloor x_i^* \rfloor$ and $x_i \geq \lceil x_i^* \rceil$, but other choices are possible.

The best choices are usually dictated by experience and perhaps the most critical choice revolves around the ability to derive good lower bounds. In practice, the branch-and-bound method often produces good solutions quickly, although in principle it could take exponential time in the worst-case.

8.3 Lagrangean Duality

We consider the integer programming problem

$$\begin{aligned} \text{minimize } & c^\top x \\ & Ax \geq b, \\ & Dx \geq d, \\ & x \in Z^n, \end{aligned} \tag{14}$$

and assume that A, D, b, c, and d have integer entries. Let Z_{IP} be the optimal cost and let

$$\mathcal{X} = \{x \in \mathbb{Z}^n \mid Dx \geq d\}.$$

The key premise behind the method is that optimizing over the set $\mathcal{X}$ can be done efficiently; for example, $\mathcal{X}$ may have an ideal characterization (which involves totally-unimodular matrices) or perhaps the combinatorial structure allows us to employ specific algorithms that work very well. However, adding the constraints $Ax \geq b$ makes the problem difficult to solve.

To that end, just like we did when we introduced the dual of an LP, we consider **relaxing these more difficult constraints and penalizing violations**. Specifically, let $p \geq 0$ be a vector of dual variables (**Lagrange multipliers**) and define

$$\begin{aligned} &\text{minimize } c^\top x + p^\top (b - Ax) \\ &x \in \mathcal{X}, \end{aligned}$$

and denote its optimal cost by $g(p)$. We will say that we *relax* or *dualize* the constraints $Ax \geq b$. For a fixed p, the above problem can be solved efficiently, as we are optimizing a linear objective over the set $\mathcal{X}$. Following our developments for LPs, we note that $g(p)$ provides a lower bound on Z_{IP}.

Lemma 1. *If the problem (14) has an optimal solution and if $p \geq 0$, then $Z(p) \leq Z_{IP}$.*

The proof is immediate and we omit it. Since this provides a lower bound for any $p \geq 0$, it is natural to consider the tightest such bound, which leads us to introduce the problem

$$\begin{aligned} &\max\ g(p) \\ &\quad p \geq 0. \end{aligned} \tag{15}$$

Mirroring our LP developments, we refer to problem (15) as the **Lagrangean dual**. Let

$$Z_D = \max_{p \geq 0} g(p).$$

Because $g(p)$ is piece-wise linear and concave (as the minimum of a finite collection of points), computing Z_D involves solving an LP with a potentially very large number of constraints. However, this leads to a lower bound and a weak duality result.

Theorem 6. *We have $Z_D \leq Z_{IP}$.*

Unlike linear programming, we do **not** have have a strong duality result. However, the most important result here is the following characterization of the Lagrangean dual.

Theorem 7. *The optimal value Z_D of the Lagrangean dual is equal to the optimal cost of the following linear programming problem:*

$$\begin{aligned} &\textit{minimize} && c^\top x \\ &\textit{subject to} && Ax \geq b, \\ &&& x \in \operatorname{conv}(\mathcal{X}). \end{aligned} \tag{16}$$

We omit the proof here and instead refer the interested reader to Bertsimas and Weismantel (2005). An important consequence of this theorem is the following ordering between the optimal value of the LP relaxation (Z_{IP}), the Lagrangean dual, and the IP:

$$Z_{\text{IP}} \leq Z_{\text{D}} \leq Z_{\text{IP}}. \tag{17}$$

The inequality on the left holds because $\text{conv}\,(\mathcal{X}) \subseteq \{x : Dx \geq d\}$, so the feasible set of problem (16) is contained in the feasible set of the LP relaxation. Problem instances exist where either of the inclusions in the relation above holds strictly (and it is also possible to have $Z_{\text{IP}} \leq Z_{\text{D}}$).

9 Dynamic Programming Methods

Lastly, we mention in passing that dynamic programming (DP) methods can also be used to solve IPs. This is readily evident if one considers the example of the discrete knapsack problem, which can be readily formulated as a DP. To understand the formulation, consider taking decisions in n stages, where n is the number of items in the knapsack. At each stage i, we decide whether to include item i in the knapsack or not. With the state variable v corresponding to **remaining volume** in the knapsack, the Bellman recursions for the problem can be written as:

$$J_i(v) = \begin{cases} J_{i+1}(v), & \text{if } v < w_j \\ \max\big(r_j + J_{i+1}(v - w_j), J_{i+1}(v)\big), & \text{otherwise.} \end{cases}$$

The recursion states that the best option in stage i (when the decision is whether to include item i or not) is to not include it in case that would exceed the remaining volume in the knapsack (i.e., if $v < w_j$) and otherwise to choose the best option between including the item and moving to stage $i+1$ with less remaining volume (which yields r_j plus the best continuation value from stage $i+1$ starting with volume $v - w_j$) or not including item i and moving to stage $i+1$ with the entire remaining item v (which yields $J_{i+1}(v)$). The recursion can be initialized at stage $i = n+1$ with $J_{n+1}(v) = 0$ for any $v \geq 0$, and solved backwards for $i = n, n-1, \ldots, 1$.

More broadly, dynamic programming techniques can actually be used to derive **pseudo-polynomial time algorithms** for solving IPs in fixed dimension, i.e., if either the number of constraints is considered fixed or the number of variables is considered fixed. Details for the interested reader are available in Schrijver (1997) and Papadimitriou and Steiglitz (1998).

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