CME 307 / MS&E 311 / OIT 676: Optimization

Interior Point Methods

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slides developed with Prof. Luiz-Rafael Santos, UFSC https://lrsantos11.github.io/

Convex optimization

Convex optimization problem

$$\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & g(x) \leq 0 \\
& Ax = b
\end{aligned}$$

where $f : \mathbf{R}^n \to \mathbf{R}$, $g : \mathbf{R}^n \to \mathbf{R}^p$ are smooth and convex, $A \in \mathbf{R}^{m \times n}$ is full rank.

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KKT conditions:

Convex optimization

Convex optimization problem

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KKT conditions:

$$\nabla f(x) + A^T y + (\nabla g(x))^T s = 0$$
$$Ax = b$$
$$g(x) \leq 0$$
$$s \geq 0$$
$$s_j g_j(x) = 0, \quad j = 1, \ldots, p$$

Outline

IPM for linear and quadratic programs

IPM for convex nonlinear programming

IPM for conic optimization

IPM for linear and quadratic programs

Linear/Quadratic Program

$$\begin{aligned}
\text{minimize} \quad & c^\top x + \tfrac{1}{2} x^\top Q x \\
\text{subject to} \quad & Ax = b, \\
& x \geq 0,
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where $Q \in \mathbf{S}_+^n$, and $A \in \mathbf{R}^{m \times n}$ is full-rank.

IPM for linear and quadratic programs

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▶ $\mathcal{P} := \{x \in \mathbf{R}^n \mid Ax = b, x \geq 0\}$ is a polyhedron.

IPM for linear and quadratic programs

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- If $Q = 0$, problem is a linear program.

IPM for linear and quadratic programs

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- If $Q = 0$, problem is a linear program.

   How to solve LP/QP problems?

IPM for linear and quadratic programs

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▶ $\mathcal{P} := \{x \in \mathbf{R}^n \mid Ax = b, x \geq 0\}$ is a polyhedron.

▶ If $Q = 0$, problem is a linear program.

How to solve LP/QP problems?

Simplex: vertex to vertex
IPM: go through the middle!

IPM for linear and quadratic programs

Linear/Quadratic Program

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How to solve LP/QP problems?

Advantages of vertex solution vs interior solution?

Simplex: vertex to vertex
IPM: go through the middle!

Building blocks of IPM

Ingredients for Interior Point Method

- ▶ Duality theory: Lagrangian function; KKT (first order optimality) condition.
- ▶ Barrier function: logarithmic barrier.
- ▶ Newton's method (and a good linear solver)

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The reward: fantastic convergence properties!

- ▶ Theoretical: $O(\sqrt{n} \log(1/\varepsilon))$ iterations
- ▶ Practical: $O(\log n \log(1/\varepsilon))$ iterations

(but the per-iteration cost may be high due to the Newton solve: often $O(n^3)$)

IPM: algorithmic template

IPM procedure

- ▶ replace inequalities with log barriers;
- ▶ form the Lagrangian;
- ▶ write down the KKT conditions of the perturbed problem;
- ▶ find one (or more) directions using Newton's method on the KKT system;
- ▶ (decide how to combine the directions and) compute a stepsize.

Duality and KKT conditions

Primal-dual QPs

Primal problem

$$\begin{array}{ll} \text{minimize} & c^\top x + \frac{1}{2} x^\top Q x \\ \text{subject to} & A x = b \\ & x \geq 0 \end{array}$$

Dual problem

$$\begin{array}{ll} \text{maximize} & b^\top y - \frac{1}{2} x^\top Q x \\ \text{subject to} & A^\top y + s - Q x = c \\ & s \geq 0 \end{array}$$

Duality and KKT conditions

Primal-dual QPs

Primal problem	**Dual problem**

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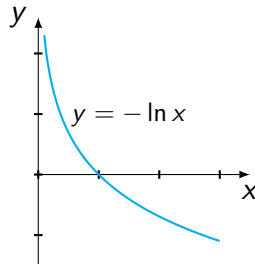## KKT conditions

$$
\begin{aligned}
Ax &= b && \rhd(\text{primal feasibility}) \\
A^\top y + s - Qx &= c && \rhd(\text{dual feasibility}) \\
XSe &= 0 && \rhd(\text{complementarity: } x_i s_i = 0, i = 1, \ldots, n) \\
(x, s) &\geq 0
\end{aligned}
$$

where $X = \mathbf{diag}(x_1, \ldots, x_n), S = \mathbf{diag}(s_1, \ldots, s_n) \in \mathbf{R}^{n \times n}$, and $e = (1, \ldots, 1) \in \mathbf{R}^n$.
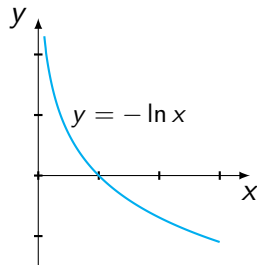
Logarithmic barrier

$-\ln x_j$
replaces the inequality
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$y = -\ln x$

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$$\text{minimize} \quad -\sum_{j=1}^{n} \ln x_j \qquad \Longleftrightarrow \qquad \text{maximize} \quad \prod_{1 \leq j \leq n} x_j$$

$\Longrightarrow$ keeps every entry of x away from 0.

Barrier primal QP

Step 1: replace inequality constraints by barrier

Replace the primal QP

$$\begin{aligned}
\text{minimize} \quad & c^\top x + \tfrac{1}{2} x^\top Q x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}$$

with the barrier primal QP

$$\begin{aligned}
\text{minimize} \quad & c^\top x + \tfrac{1}{2} x^\top Q x - \mu \sum_{j=1}^{n} \ln x_j \\
\text{subject to} \quad & Ax = b
\end{aligned}$$

Logarithmic barrier and stationarity

Step 2: remove equality constraints using Lagrangian

$$\mathcal{L}(x, y, \mu) = c^\top x + \frac{1}{2} x^\top Q x - y^\top (Ax - b) - \mu \sum_{j=1}^{n} \ln x_j$$

Logarithmic barrier and stationarity

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A stationary point (x, y, μ) of the Lagrangian satisfies

$$\nabla_x \mathcal{L}(x, y, \mu) = 0 \qquad\qquad = c + Qx - A^\top y - \mu X^{-1} e$$

with $X^{-1} = \mathbf{diag}(x_1^{-1}, \ldots, x_n^{-1}) \in \mathbf{R}^{n \times n}, (x_j > 0)$.

KKT conditions for barrier problem

▶ Define $s := \mu X^{-1}e$, which implies $XSe = \mu e$, to get

KKT_μ

$$Ax = b$$
$$A^\top y + s - Qx = c$$
$$XSe = \mu e$$
$$(x, s) > 0$$

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KKT$_\mu$ KKT

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$$\text{KKT}_\mu \to \text{KKT as } \mu \to 0.$$

Central path (LP case)

▶ Parameter μ controls the distance to optimality

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▶ Analytic center (μ-center): unique point

$$(x(\mu), y(\mu), s(\mu)), \qquad x(\mu) > 0, \ s(\mu) > 0$$

that satisfies the KKT_μ conditions.

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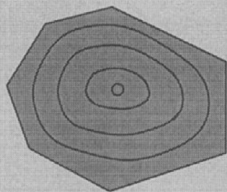
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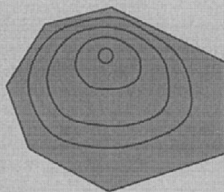
that satisfies the KKT_μ conditions.

▶ The curve

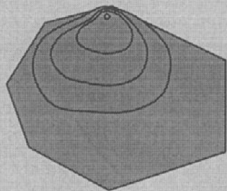$$\mathcal{C}_\mu = \{(x(\mu), y(\mu), s(\mu)) \mid \mu > 0\}$$
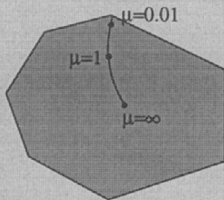
is called the primal-dual central path.

(a) μ=∞

(b) μ=1

(c) μ=0.01

(d) central path

Recall Newton's method for nonlinear equation

- For $F : \mathbf{R}^n \to \mathbf{R}^n$ smooth, solve $F(x) = 0$.

Recall Newton's method for nonlinear equation

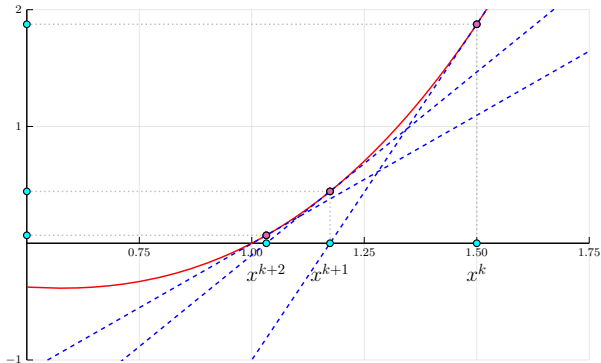- For $F : \mathbf{R}^n \to \mathbf{R}^n$ smooth, solve $F(x) = 0$.
- Newton's method: define Jacobian $J_F(x)$ so $J_F(x)_{ij} = \frac{\partial F_i}{\partial x_j}$, and iterate

$$x^{k+1} = x^k - \alpha_k J_F(x^k)^{-1} F(x^k)$$

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Apply Newton Method to KKT$_\mu$

The first order optimality conditions for the barrier problem form a large system of nonlinear equations:

$$F(x, y, s) = 0,$$

where $F : \mathbf{R}^{2n+m} \mapsto \mathbf{R}^{2n+m}$ is defined as

$$F(x, y, s) = \begin{bmatrix} Ax & -b \\ A^\top y + s - Qx & -c \\ XSe & -\mu e \end{bmatrix}$$

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▶ The first two blocks are linear.
▶ The last block, corresponding to the complementarity condition, is nonlinear.
▶ Jacobian is

$$J_F(x, y, s) = \begin{bmatrix} A & 0 & 0 \\ -Q & A^\top & I \\ S & 0 & X \end{bmatrix}$$

Interior-point QP Algorithm

IPM Framework

Fix the barrier parameter μ and make *one* (damped) Newton step towards the solution of KKT_μ. Then reduce the barrier parameter μ and repeat.

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▶ For $k = 1, 2, \ldots$

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  - $\mu_k = \sigma \mu_{k-1}$, where $\sigma \in (0, 1)$

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    - Find Newton direction $(\Delta x^k, \Delta y^k, \Delta s^k)$ by solving

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- Find step length α_k so $(x^k + \alpha_k \Delta x^k, y^k + \alpha_k \Delta y^k, s^k + \alpha_k \Delta s^k)$ is feasible.

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  - Find step length α_k so $(x^k + \alpha_k \Delta x^k, y^k + \alpha_k \Delta y^k, s^k + \alpha_k \Delta s^k)$ is feasible.
  - Make step $(x^{k+1}, y^{k+1}, s^{k+1}) = (x^k + \alpha_k \Delta x^k, y^k + \alpha_k \Delta y^k, s^k + \alpha_k \Delta s^k)$.

Path-following algorithm

▶ **Short-step path-following method:** $\mathcal{O}(\sqrt{n})$ complexity result

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Theorem ([Gondzio, 2012, Thm. 3.1])

Given $\epsilon > 0$, suppose that a feasible starting point $(x^0, y^0, s^0) \in \mathcal{N}_2(0.1)$ satisfies

$$\left(x^0\right)^\top s^0 = n\mu^0, \text{ where } \mu^0 \leq 1/\epsilon^\kappa,$$

for some positive constant κ. Then for some $K = \mathcal{O}(\sqrt{n}\ln(1/\epsilon))$, the optimality gap is bounded by ϵ after at most K iterations:

$$\mu^k \leq \epsilon, \quad \forall k \geq K$$

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▶ θ-neighborhood of the central path:
$\mathcal{N}_2(\theta) := \{(x, y, s) \in \mathcal{F}^0 \mid \|XSe - \mu e\| \le \theta\mu\}$, with $\mu = \frac{1}{n}x^\top s$.

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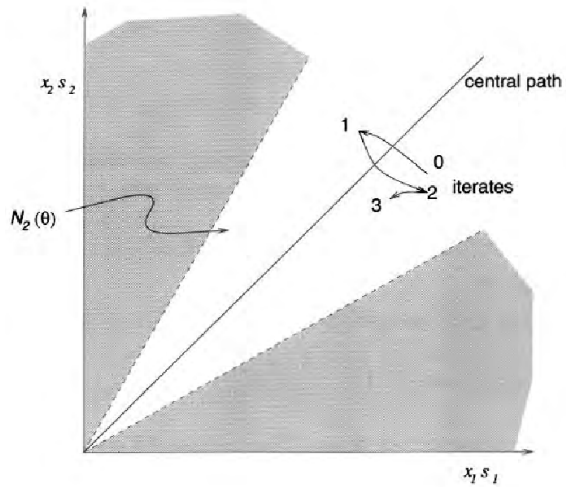*for some positive constant κ. Then for some $K = \mathcal{O}(\sqrt{n}\ln(1/\epsilon))$, the optimality gap is bounded by ϵ after at most K iterations:*

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▶ Slow progress towards optimality

Augmented system

Newton direction

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^\top & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^\top y - s + Qx \\ \mu_k e - XSe \end{bmatrix} =: \begin{bmatrix} \xi_p \\ \xi_d \\ \xi_\mu \end{bmatrix}$$

use last (complementarity) block to solve for Δs as a function of Δx.

Augmented system

Define $\Theta = XS^{-1}$ (ill-conditioned!). Then Δx and Δy solve the Newton system
$\iff$

$$\begin{bmatrix} -Q - \Theta^{-1} & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \xi_d - X^{-1}\xi_\mu \\ \xi_p \end{bmatrix}$$

▶ Newton system is nonsymmetric.
▶ Augmented system is symmetric but indefinite.

Normal equations

Augmented system

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Normal equations

Eliminate Δx to arrive at the *Normal equations*

$$(A\Theta A^\top)\Delta y = A\Theta g + \xi_p$$

Normal equations

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Normal equations

Eliminate Δx to arrive at the *Normal equations*

$$(A\Theta A^\top)\Delta y = A\Theta g + \xi_p$$

▶ $A\Theta A^\top$ is symmetric and positive semidefinite. (Finally!)
▶ Normal equations in QP $(A(Q + \Theta)A^\top)\Delta y = g$ are generally nearly dense, even when A and Q are sparse.
▶ LP: Normal equations are often used.
▶ QP: usually use the indefinite augmented system.

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▶ Convex NLP

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IPM for NLP

▶ Convex NLP

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) \le 0 \end{array} \iff \begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g(x) + z = 0, \quad z \ge 0 \end{array}$$

▶ Replace inequality $z \ge 0$ with logarithmic barrier

$$\text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(z_i) \quad \text{subject to} \quad g(x) + z = 0$$

IPM for NLP

▶ Convex NLP

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▶ Replace inequality $z \geq 0$ with logarithmic barrier

$$
\text{minimize} \quad f(x) - \mu \sum_{i=1}^{m} \ln(z_i) \quad \text{subject to} \quad g(x) + z = 0
$$

▶ Write out Lagrangian

$$
L(x, y, z, \mu) = f(x) + y^\top (g(x) + z) - \mu \sum_{i=1}^{m} \ln(z_i)
$$

IPM for NLP

▶ Write conditions for stationary point

$$\nabla_x L(x, z, y) = \nabla f(x) + J_g(x)^\top y = 0$$
$$\nabla_y L(x, z, y) = g(x) + z = 0$$
$$\nabla_z L(x, z, y) = y - \mu Z^{-1} e = 0$$

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▶ Write KKT system

$$\nabla f(x) + J_g(x)^\top y = 0,$$
$$g(x) + z = 0$$
$$YZe = \mu e$$

Newton for KKT of NLP

▶ Apply Newton method for KKT system

Newton for KKT of NLP

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▶ Jacobian matrix of KKT system

$$J_F(x, z, y) = \begin{bmatrix} Q(x, y) & J_g(x)^\top & 0 \\ J_g(x) & 0 & I \\ 0 & Z & Y \end{bmatrix}$$

where $Q(x, y) = \nabla^2 f(x) + \sum_{i=1}^m y_i \nabla^2 g_i(x)$ is the Hessian of L

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▶ Newton step for KKT system

$$\begin{bmatrix} Q(x,y) & J_g(x)^\top & 0 \\ J_g(x) & 0 & I \\ 0 & Z & Y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} -\nabla f(x) - J_g(x)^\top y \\ -g(x) - z \\ \mu e - YZe \end{bmatrix}$$

From QP to NLP

▶ Newton direction for NLP

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▶ Need to compute $Q(x,y)$ and $J_g(x)$ at each iteration
▶ Caveat: use trust region method to choose stepsize as Hessian may be indefinite.

Outline

Self-concordant function

Definition

Function f is *self-concordant* if for some constant $M_f \geq 0$, the inequality

$$\nabla^3 f(x)[u, u, u] \leq M_f \|u\|_{\nabla^2 f(x)}^{3/2}$$

holds for any $x \in \text{dom}\, f$ and $u \in R^n$.

▶ A self-concordant function is always well approximated by a quadratic model because the error of such an approximation can be bounded by the $\|u\|_{\nabla^2 f(x)}^{3/2}$

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Theorem ([Boyd and Vandenberghe, 2004, Section 11.5])

Newton's method with line search finds an ε approximate solution in less than $T := \text{constant} \times (f(x_0) - f^\star) + \log_2 \log_2 \frac{1}{\varepsilon}$ iterations.

Log-barrier is self-concordant

Theorem

The barrier function $-\ln(x)$ is self-concordant in $\mathbf{R}_+$.

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Consider $f(x) = -\ln(x)$. This function is self-concordant in $\mathbf{R}_+$ because

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Compute and check that self-concordance holds with $M_f = 2$. $\qquad\square$

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Conic optimization

▶ Consider the optimization problem

$$
\begin{aligned}
\text{minimize} \quad & c^\top x \\
\text{subject to} \quad & Ax = b \\
& x \in K
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where K is a convex closed cone.

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▶ Conic optimization can be solved in polynomial time with IPMs

Second-order conic optimization

- $K = \mathbb{L} := \{(x,t) \mid x \in \mathbf{R}^{n-1}, t \in \mathbf{R}, \|x\|_2 \leq t, t \geq 0\}$ (Second-order cone)

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$$f(x, t) = \begin{cases} -\ln(t^2 - \|x\|_2^2) & \text{if } \|x\| < t \\ +\infty & \text{otherwise} \end{cases}$$

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Exercise: Prove in case $n = 2$.

Semidefinite programming

▶ Variable now is a symmetric matrix $X \in K = \mathbf{S}^n$

SDP and its dual

minimize $\quad C \bullet X$

subject to $\quad A_i \bullet X = b_i, i = 1, \ldots, m$

$\qquad\qquad X \succeq 0$

maximize $\quad b^\top y$

subject to $\quad \sum_{i=1}^m y_i A_i + S = C$

$\qquad\qquad S \succeq 0$

A_i, $C \in \mathbf{S}^n$ and $b \in \mathbf{R}^m$ given, and $X, S \in \mathbf{S}^n$ and $y \in \mathbf{R}^m$ unknown.

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- Variable now is a symmetric matrix $X \in K = \mathbf{S}^n$
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A_i, $C \in \mathbf{S}^n$ and $b \in \mathbf{R}^m$ given, and $X, S \in \mathbf{S}^n$ and $y \in \mathbf{R}^m$ unknown.

Theorem (Weak duality for SDP)

If X is primal feasible and (y, S) is dual feasible, then

$$C \bullet X - b^\top y = X \bullet S \geq 0$$

Logarithmic barrier for SDP

▶ Logarithmic barrier function for the semi-definite cone

$$f(X) = \begin{cases} -\ln(\det(X)) & \text{if } X \succ 0 \\ +\infty & \text{otherwise} \end{cases}$$

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▶ Let $X \succ 0$ and $H \in \mathbf{S}^n$ be given. Then

$$\begin{aligned}
f(X + tH) &= -\ln(\det(X + tH)) = -\ln(\det(X(I + tX^{-1}H))) \\
&= -\ln(\det(X)) - \ln(\det(I + tX^{-1}H)) \\
&= -\ln(\det(X)) - \ln(1 + t\,\text{tr}(X^{-1}H) + \mathcal{O}(t^2)) \\
&= f(X) - tX^{-1} \bullet H + \mathcal{O}(t^2)
\end{aligned}$$

Derivatives of Logarithmic barrier for SDP

▶ First derivative of $f(X)$

$$f'(X) = \lim_{t \to 0} \frac{f(X + tH) - f(X)}{t} = -X^{-1}$$

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So $Df(X)[H] = -X^{-1} \bullet H$.

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▶ Second derivative of $f(X)$

$$f'(X + tH) = -[X(I + tX^{-1}H)]^{-1} = -[I - tX^{-1}H + \mathcal{O}(t^2)]X^{-1}$$
$$= f'(X) + tX^{-1}HX^{-1} + \mathcal{O}(t^2)$$

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$$= f'(X) + tX^{-1}HX^{-1} + \mathcal{O}(t^2)$$

so $f''(X)[H] = X^{-1}HX^{-1}$ and $D^2f(X)[H, G] = X^{-1}HX^{-1} \bullet G$.

▶ $f'''(X)[H, G] = -X^{-1}HX^{-1}GX^{-1} - X^{-1}GX^{-1}HX^{-1}$

Characterization of self-concordance for SDP

Theorem

The function $f(X) = -\ln \det X$ is a convex barrier for $\mathbf{S}^n_+$.

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Proof sketch.

Let $\varphi(t) = F(X + tH)$. Then, prove that $\varphi''(t) \geq 0$ for $t > 0$ such that $X + tH \succ 0$. Therefore, when $X \succ 0$ approaches a singular matrix, its determinant approaches zero, and the function $f(X) \to +\infty$. $\qquad \square$

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Theorem ([Nestervov and Nemirovskii, 1994])

The barrier function $f(X) = -\ln \det X$ is self-concordant on $\mathbf{S}_+^n$.

Solving SDPs with IPMs

▶ Replace the primal SDP

$$\begin{aligned}
\text{minimize} \quad & C \bullet X \\
\text{subject to} \quad & \mathcal{A}X = b, \\
& X \succeq 0,
\end{aligned}$$

with the primal barrier SDP

$$\begin{aligned}
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(with a barrier parameter $\mu \geq 0$).

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▶ Formulate the Lagrangian

$$L(X, y, S) = C \bullet X + \mu f(X) - y^T(\mathcal{A}X - b),$$

with $y \in \mathcal{R}^m$, and write the first order conditions (FOC) for a stationary point of L:

$$C + \mu f'(X) - \mathcal{A}^* y = 0$$

Solving SDPs with IPMs (cont'd)

▶ Use $f(X) = -\ln \det X$ and $f'(X) = -X^{-1}$ to obtain

$$C - \mu X^{-1} - \mathcal{A}^* y = 0$$

Solving SDPs with IPMs (cont'd)

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$$C - \mu X^{-1} - \mathcal{A}^* y = 0$$

▶ Denote $S = \mu X^{-1}$, i.e., $XS = \mu I$. Then, the FOC can be written as

$$\mathcal{A}X = b$$
$$A^* y + S = C$$
$$XS = \mu I$$

with $X, S \in \mathbf{S}^n_{++}$.

Newton direction

Differentiating this system is hard! The Newton direction solves:

$$
\begin{bmatrix}
\mathcal{A} & 0 & 0 \\
0 & \mathcal{A}^* & \mathcal{I} \\
\mu \left(X^{-1} \odot X^{-1} \right) & 0 & \mathcal{I}
\end{bmatrix}
\cdot
\begin{bmatrix}
\Delta X \\
\Delta y \\
\Delta S
\end{bmatrix}
=
\begin{bmatrix}
\xi_b \\
\xi_C \\
\xi_\mu
\end{bmatrix}.
$$

We define the Kronecker product $P \odot Q$ for $P, Q \in \mathbf{R}^{n \times n}$, which yields a linear operator from $\mathbf{S}^n$ to $\mathbf{S}^n$ given by

$$
(P \odot Q)U = \frac{1}{2} \left(PUQ^T + QUP^T \right).
$$

Summary

▶ IPM for SOCP and SDP with self-concordant barrier:

source: [Gondzio, 2012: Interior Point Methods 25 Years Later]

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Summary

- ▶ IPM for SOCP and SDP with self-concordant barrier:
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