MODELING WITH BINARY VARIABLES

Class 3 – October 1, 2025

Context

- You have several projects available A, B, ...,

- You choose **which projects to fund**

- A=1 **if and only if** project A is funded

If you fund **A**, you should also fund **E**

- What are the feasible values for A, E?
  - Recall that A, E are **binary**
  - We want: *if A=1, must have E=1*

ALL OPTIONS:

A	E	
0	0	✓
0	1	✓
~~1~~	~~0~~	
1	1	✓

- How about: **A** $\leq$ **E**
  - If A=1, the only option is E=1
  - If A=0, can set any value for E

- **Remember!** "If you fund **A**, then you should fund **B**": **A** $\leq$ **B**

- **Q:** "If you do **not** fund **A**, then you should fund **B**"
  - Add a constraint: $1 - A \leq B$
    - "Not selecting A" is same as $1 - A = 1$, so this is just like **Q5 !**

Logical Implications with Binary Variables

- **Q**. If you fund project A, then you should fund projects E **and** H.
  - Same as: “If you fund A, then fund E” and “If you fund A, then fund H”
  - A <= E, A <= H
  - Also possible to do this with **one** constraint: A <= (E+H)/2

  Q. Why not A <= E+H?

- **Q**. If you fund anything from **A/B/C**, then also fund **H**.
  - Same as: “If you fund A, then fund H” and “If you fund B, then fund H”, …
  - A <= H, B <= H, C <= H
  - Also possible to do this with **one** constraint: (A+B+C)/3 <= H

  Q. Why not A + B + C <= H?

General Recipe for Defining Indicators

$$\mathbf{Y = 1} \textit{ if and only if } \mathbf{a_1 X_1 + \ldots + a_n X_n + b \geq 0}$$

- **Y** is a binary decision variable, $X_1, \ldots, X_n$ are continuous or discrete decisions
- $a_1, \ldots, a_n, b$ are parameters/data

- **The first implication:**

(1): *If* **Y = 1** *then* $a_1 X_1 + \ldots + a_n X_n + b \geq 0$

- This is equivalent to the following linear constraint:

$$a_1 X_1 + \ldots + a_n X_n + b \geq m \cdot (1 - Y)$$

  - *In practice, '**m**' is the smallest value that* $a_1 X_1 + \ldots + a_n X_n + b$ *can take*

- **Understand why this works. No need to remember the constraint!**

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- **The first implication:**

  (1): *If* **Y = 1** *then* $a_1 X_1 + \ldots + a_n X_n + b \geq 0$

- In practice, you can directly implement **(1)** in Gurobi with:

  model.addGenConstrIndicator(Y, ***True****,* $a_1 X_1 + \ldots + a_n X_n + b \geq 0$ *)*

  Syntax: *model.addGenConstrIndicator(* **Y***,* **boolean value***,* **implied (in)equality***)*

  - **Y** = a Gurobi binary variable
  - **boolean value** = True or False
  - **implied (in)equality** = linear relationship that should hold when Y = **boolean value**

  This implements **one** direction: *"If Y=boolean value, then implied (in)equality"*
  https://www.gurobi.com/documentation/current/refman/py_model_agc_indicator.html

General Recipe for Defining Indicators

$$\mathbf{Y = 1} \textit{ if and only if } \mathbf{a_1 X_1 + \ldots + a_n X_n + b \geq 0}$$

- **Y** is a binary decision variable, $X_1, \ldots, X_n$ are continuous or discrete decisions
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- **The second implication:**

(2) *If* $Y = 0$ *then* $a_1 X_1 + \ldots + a_n X_n + b < 0$

- Because we cannot have **strict** inequality < **0**, instead we implement:

If $Y = 0$ *then* $a_1 X_1 + \ldots + a_n X_n + b \leq -\varepsilon$

  - If $X_1, \ldots, X_n$ are integer, reformulation can be made exact. Otherwise, take '**ε**' as a small tolerance (e.g., 0.00001).

- Implemented with: $a_1 X_1 + \ldots + a_n X_n + b + \varepsilon \leq (M + \varepsilon)\, Y$
  - *In practice, '**M**' is the largest value that* $a_1 X_1 + \ldots + a_n X_n + b$ *can take*

Recap

$$Y = 1 \textit{ if and only if } a_1 X_1 + \ldots + a_n X_n + b \geq 0$$

Y is a binary decision variable, $X_1, \ldots, X_n$ are continuous or discrete decisions

$a_1, \ldots, a_n, b$ are parameters/data

(1): *If* $Y = 1$ *then* $a_1 X_1 + \ldots + a_n X_n + b \geq 0$

(2): *If* $Y = 0$ *then* $a_1 X_1 + \ldots + a_n X_n + b < 0$

Implemented with linear constraints:

(1) $a_1 X_1 + \ldots + a_n X_n + b \geq m \cdot (1 - Y)$

(2) $a_1 X_1 + \ldots + a_n X_n + b + \varepsilon \leq (M + \varepsilon)\, Y$ (ε=1 if $X_1, \ldots, X_n$ integer)

In Gurobi:

(1) *model.addGenConstrIndicator(* Y, *True*, $a_1 X_1 + \ldots + a_n X_n + b \geq 0$ *)*

(2) *model.addGenConstrIndicator(* Y, *False*, $a_1 X_1 + \ldots + a_n X_n + b \leq -\varepsilon$ *)*

"Cheat-Sheet"

X and **Y** are decisions; **a**, **b** are parameters/data; **a X** denotes any linear expression in **X**

1. (X,Y bin) ***"If X = 1 then Y = 1"*** → add constraint: **X ≤ Y**
2. (X,Y bin) ***"If X = 1 then Y = 1, and vice-versa"*** → add constraint: **X = Y**

3. (Y bin) ***"If Y = 1 then a X + b ≥ 0"*** → add constraint: ***a X + b ≥ m ·(1-Y)***
   - 'm' is the *smallest* value *a X + b* can take
4. (Y bin) ***"If Y = 1 then a X ≥ b"*** → add constraint: ***a X – b ≥ m · (1-Y)***
   - 'm' is the *smallest* value *(a X – b)* can take
5. (Y bin) ***"If Y = 1 then a X ≤ b "*** → add constraint: ***a X – b ≤ M · (1-Y)***
   - 'M' is the *largest* value *(a X - b)* can take
6. (Y bin) ***"If Y = 1 then a X + b ≤ 0 "*** → add constraint: ***a X + b ≤ M · (1-Y)***
   - 'M' is *largest* value (*a X + b*) can take

3-6 are all "the same"! Use whichever you like!

7. (Y bin) ***"If Y = 1 then a X + b > 0 "*** → **CAN'T DO > 0.**
   - Instead, do ***"If Y = 1 then a X + b ≥ ε" for a very small number ε > 0***
   - To implement, add the constraint: ***a X + b – ε ≥ (m – ε) (1-Y),*** where 'm' is the smallest value (*a X + b*) can take

8. If you need ***"If Y = 0 then …"***, replace ***Y*** in the constraint with ***1-Y***
9. If you need ***"If a X + b ≤ 0 then Y = 1"***, replace this with ***"If Y=0, then a X + b > 0"***

10. (Y bin) Need **"X * Y"** → add new variable **Z** ("= X * Y") and constraints:

    Z ≤ M · Y **Z ≥ m · Y** **Z ≤ X – m · (1 – Y)** **Z ≥ X – M · (1 – Y)**
    - m/M are smallest/largest value that X can take

Duality

Lecture 4

October 1, 2025

Motivation

Consider an optimization problem

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   Formally, how to quantify the gap $c^\mathsf{T} x - p^\star$ where $p^\star$ is the optimal value?

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2. *Without a feasible x, how to* **certify** *that $\{x : Ax \leq b\}$ is empty?*
3. Suppose one constraint is: $a_i^{\mathsf{T}} x \leq 0$ where $a_i \in \mathcal{A}$ are unknown parameters. *How to find an x that is feasible* **for any** *$a_i \in \mathcal{A}$?*

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2. *Without a feasible x, how to* **certify** *that $\{x : Ax \leq b\}$ is empty?*
3. Suppose one constraint is: $a_i^{\mathsf{T}} x \leq 0$ where $a_i \in \mathcal{A}$ are unknown parameters. *How to find an x that is feasible* **for any** $a_i \in \mathcal{A}$*?*
4. You are offered a bit more of b_i, for a "suitable price". *Is the deal worthwhile?*

Duality theory will provide answers to these questions (and more)

Outline

- Consider a **primal** optimization problem:

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- We will show that the dual provides lower bounds for the primal:

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$$c^\mathsf{T} x^\star = \tilde{r}^\mathsf{T} y^\star \quad \textbf{(strong duality)}$$

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- In the process, will uncover some **fundamental ideas in optimization**:

separation of convex sets $\Longrightarrow$ **Farkas Lemma** $\Longrightarrow$ **strong duality**

Deriving Lower Bounds

Consider a linear optimization problem in the most general form possible:

Primal Problem

$$
\begin{aligned}
(\mathcal{P})\ \text{minimize}_x \quad & c^{\mathsf{T}}x \\
\text{such that} \quad & a_i^{\mathsf{T}}x \geq b_i, && \forall i \in I_{\text{ge}}, \\
& a_i^{\mathsf{T}}x \leq b_i, && \forall i \in I_{\text{le}}, \\
& a_i^{\mathsf{T}}x = b_i, && \forall i \in I_{\text{eq}}, \\
& x_j \geq 0, && \forall j \in J_p, \\
& x_j \leq 0, && \forall j \in J_n, \\
& x_j \ \text{free}, && \forall j \in J_f \\
\text{variable} \quad & x \in \mathbb{R}^n.
\end{aligned} \tag{1}
$$

Note the mnemonic encoding...

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\end{aligned}
\qquad
A = \begin{bmatrix} A^{\text{le}} \\ A^{\text{ge}} \\ A^{\text{eq}} \end{bmatrix} \in \mathbb{R}^{m \times n}
\qquad
b = \begin{bmatrix} b^{\text{le}} \\ b^{\text{ge}} \\ b^{\text{eq}} \end{bmatrix} \in \mathbb{R}^{m}
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& a_i^T x = b_i, & \forall i \in I_{\text{eq}}, & \\
& x_j \geq 0, & \forall j \in J_p, & \\
& x_j \leq 0, & \forall j \in J_n, & b = \begin{bmatrix} b^{\text{le}} \\ b^{\text{ge}} \\ b^{\text{eq}} \end{bmatrix} \in \mathbb{R}^m \\
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\text{variable} & x \in \mathbb{R}^n. & &
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Definition

We will refer to this as the **primal problem** or problem $(\mathcal{P})$.

Let P denote its feasible set (a polyhedron), and $p^\star$ denote its optimal value.

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$(\mathcal{P})$ is a minimization; we seek **valid lower bounds** on $(\mathcal{P})$. *Any ideas?*

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Can **remove** constraints! Drastic, and could end up with a bound of $-\infty$!

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Let's **relax** some constraints and penalize ourselves for the relaxation! *Which / how?*

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General principle: (i) relax "complicating" constraints; (ii) try "simple" penalty

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For every constraint i, have a **penalty** λ_i

Construct the **lower bound** as the **Lagrangean**:

$$\mathcal{L}(x, \lambda) = c^\mathsf{T} x - \sum_{i=1}^{m} \lambda_i (a_i^\mathsf{T} x - b_i) = c^\mathsf{T} x - \lambda^\mathsf{T} (Ax - b)$$

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Note: we relaxed the complicating constraints, $a_i^\mathsf{T} x \; \textcircled{?} \; b_i$, and used a linear penalty

Not apriori clear that this will give us very good bounds...

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We want the Lagrangean to give us **a valid lower bound**:

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\lambda_i \text{ free}, & \forall i \in I_{\text{eq}}.
\end{array}
\right\} \Leftrightarrow \lambda \in \Lambda \tag{2}
$$

Deriving Lower Bounds

Summarizing... any $\lambda \in \Lambda$ produces **a valid lower bound**:

$$\mathcal{L}(x, \lambda) = c^{\mathsf{T}} x - \lambda^{\mathsf{T}}(Ax - b) \leq c^{\mathsf{T}} x, \; \forall x \in P.$$

How can we get a lower bound on the primal's **optimal value** $p^\star$*?*

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How can we get a lower bound on the primal's **optimal value** *$p^\star$?*

Claim

The function $g : \Lambda \to \mathbb{R}$ defined as:

$$\begin{aligned} g(\lambda) := min_x \; & \mathcal{L}(x, \lambda) \\ s.t. \; & x_j \geq 0, \; \forall j \in J_p \\ & x_j \leq 0, \; \forall j \in J_n \\ & x_j \; free, \; \forall j \in J_f \end{aligned} \tag{3}$$

satisfies $g(\lambda) \leq p^\star$ for any $\lambda \in \Lambda$.

Note: including the sign constraints on x in this optimization improves the lower bound!

Deriving Lower Bounds

Let us analyze this further:

$$
\begin{aligned}
g(\lambda) = \min_x \ & \mathcal{L}(x, \lambda) & = \min_x \ & \left[\lambda^\mathsf{T} b + (c^\mathsf{T} - \lambda^\mathsf{T} A)x\right] \\
\text{s.t. } & x_j \geq 0, \ \forall j \in J_p, & \text{s.t. } & x_j \geq 0, \ \forall j \in J_p, \\
& x_j \leq 0, \ \forall j \in J_n, & & x_j \leq 0, \ \forall j \in J_n, \\
& x_j \text{ free}, \ \forall j \in J_f & & x_j \text{ free}, \ \forall j \in J_f
\end{aligned}
$$

Deriving Lower Bounds

Let us analyze this further:

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x_j \text{ free}, \ \forall j \in J_f \qquad & x_j \text{ free}, \ \forall j \in J_f
\end{aligned}
$$

$$
g(\lambda) = \begin{cases} \lambda^{\mathsf{T}} b, & \text{if } \lambda^{\mathsf{T}} A_j \leq c_j, \forall j \in J_p \ \textbf{and} \ \lambda^{\mathsf{T}} A_j \geq c_j, \forall j \in J_n \ \textbf{and} \ \lambda^{\mathsf{T}} A_j = c_j, \forall j \in J_f \\ -\infty, & \text{otherwise.} \end{cases}
$$

Deriving the Dual Problem

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is **a valid lower bound** on the primal **optimal value**: $g(\lambda) \le p^\star$ for any $\lambda \in \Lambda$.

How can we get the **best** *lower bound?*

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How can we get the **best** *lower bound?*

$$\underset{\lambda \in \Lambda}{\text{maximize}}\, g(\lambda) \tag{4}$$

This is equivalent to the following optimization problem:

Deriving the Dual Problem

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This is equivalent to the following optimization problem:

Dual Problem

$$\begin{array}{rlr} \text{maximize} & \lambda^{\mathsf{T}} b & \\ \text{subject to} & \lambda_i \geq 0, & \forall i \in I_{\text{ge}}, \\ & \lambda_i \leq 0, & \forall i \in I_{\text{le}}, \\ & \lambda_i \text{ free}, & \forall i \in I_{\text{eq}}, \\ & \lambda^{\mathsf{T}} A_j \leq c_j, & \forall j \in J_p, \\ & \lambda^{\mathsf{T}} A_j \geq c_j, & \forall j \in J_n, \\ & \lambda^{\mathsf{T}} A_j = c_j, & \forall j \in J_f. \end{array} \tag{5}$$

Deriving the Dual Problem

Dual Problem

$$
\begin{aligned}
\text{maximize} \quad & \lambda^{\mathsf{T}} b \\
\text{subject to} \quad & \lambda_i \geq 0, && \forall i \in I_{\text{ge}}, \\
& \lambda_i \leq 0, && \forall i \in I_{\text{le}}, \\
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\end{aligned} \tag{6}
$$

Deriving the Dual Problem

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\end{aligned} \tag{6}
$$

Definition

This is the **dual** of $(\mathcal{P})$, which we will also refer to as $(\mathcal{D})$. We denote its feasible set with D and its optimal value with $d^\star$.

Note: The dual is also a linear optimization problem!

Primal-Dual Pair

Primal-Dual Pair of Problems

Primal $(\mathcal{P})$

$$
\begin{array}{rlll}
\underset{x}{\text{minimize}} & c^{\mathsf T} x & & \\
(\lambda_i \rightarrow) & a_i^{\mathsf T} x \geq b_i, & \forall i \in I_{\text{ge}} \\
(\lambda_i \rightarrow) & a_i^{\mathsf T} x \leq b_i, & \forall i \in I_{\text{le}} \\
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 & x_j \text{ free}, & \forall j \in J_f \\
\text{variables} & x \in \mathbb{R}^n & &
\end{array}
$$

Dual $(\mathcal{D})$

$$
\begin{array}{rll}
\underset{\lambda}{\text{maximize}} & \lambda^{\mathsf T} b & \\
 & \lambda_i \geq 0, & \forall i \in I_{\text{ge}} \\
 & \lambda_i \leq 0, & \forall i \in I_{\text{le}} \\
 & \lambda_i \text{ free}, & \forall i \in I_{\text{eq}} \\
 & \lambda^{\mathsf T} A_j \leq c_j, & \forall j \in J_p \\
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 & \lambda^{\mathsf T} A_j = c_j, & \forall j \in J_f \\
\text{variables} & \lambda \in \mathbb{R}^m. &
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\end{array}
$$

Recall the procedure for deriving the dual:

- a dual decision variable λ_i for every primal constraint (except variable signs)
- constrain λ_i to ensure lower bound: λ_i ⓘ 0
- for every primal decision x_j, add a dual constraint in the form $\lambda^{\mathsf{T}} A_j$? c_j (involving the column A_j and the objective coefficient c_j corresponding to λ_i)

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Exercise

Rewrite the dual problem as a minimization problem and construct its dual.

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Exercise

Rewrite the dual problem as a minimization problem and construct its dual.

Theorem (For LPs, the dual of the dual is the primal)

If we transform the dual of a linear optimization problem into an equivalent minimization problem and form its dual, we obtain a problem equivalent to the primal.

Rules for Constructing the Dual of Any LP

Consider any linear optimization problem (minimization/maximization):

$$\begin{aligned} \underset{x}{\text{minimize}} \;/\; \underset{x}{\text{maximize}} \quad & c^{\mathsf{T}} x \\ (\lambda \rightarrow) \quad & Ax \begin{smallmatrix}\leq \\ = \\ \geq\end{smallmatrix} b \\ & x \begin{smallmatrix}\leq \\ = \\ \geq\end{smallmatrix} 0 \end{aligned} \tag{7}$$

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R1: A dual variable λ_i for every constraint, i.e., every row a_i^T of A.
λ_i **free** for **equality** constraints $(a_i^\mathsf{T} x = b_i)$. Otherwise: λ_i ⓘ 0.

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R2: In the dual, add a constraint for *every primal variable* x_j
If x_j is **free**, write this as $\lambda^{\mathsf{T}} A_j = c_j$. Otherwise: $\lambda^{\mathsf{T}} A_j$? c_j.

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R3: To determine the signs (?), use this rule of thumb:

the dual variable λ_i is the (sub)gradient of the optimal objective value with respect to the constraint's right-hand-side b_i

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the dual variable λ_i is the (sub)gradient of the optimal objective value with respect to the constraint's right-hand-side b_i

- in a minimization, for a "$\leq$" constraint, the dual variable is ≤ 0
- in a minimization, for a "$\geq$" constraint, the dual variable is ≥ 0
- in a maximization, for a "$\leq$" constraint, the dual variable is ≥ 0
- in a maximization, for a "$\geq$" constraint, the dual variable is ≤ 0.

Example 1

$$
\begin{aligned}
(\mathcal{P}) \quad & \max\ 3x_1 + 2x_2 & \\
& \text{s.t. } x_1 + 2x_2 \le 4 & (1) \\
& 3x_1 + 2x_2 \ge 6 & (2) \\
& x_1 - x_2 = 1 & (3) \\
& x_1,\ x_2 \ge 0. &
\end{aligned}
$$

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(\mathcal{P}) \quad & \max\ 3x_1 + 2x_2 \\
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& \qquad x_1,\ x_2 \ge 0.
\end{aligned}
$$

$$
\begin{aligned}
(\mathcal{D}) \quad & \min\ 4y_1 + 6y_2 + y_3 \\
& \text{s.t. } y_1 + 3y_2 + y_3 \ge 3, \\
& \qquad 2y_1 + 2y_2 - y_3 \ge 2, \\
& \qquad y_1 \ge 0, \quad y_2 \le 0, \quad y_3 \text{ free.}
\end{aligned}
$$

Some Quick Results

Theorem ("Duals of equivalent primals")

If we transform a primal P_1 into an equivalent formulation P_2 by:

- *replacing a free variable x_i with $x_i = x_i^+ - x_i^-$,*
- *replacing an inequality with an equality by introducing a slack variable,*
- *removing linearly dependent rows $a_i^\top$ for a* **feasible** *LP in standard form,*

then the duals of (P_1) and (P_2) are **equivalent***, i.e., they are either both infeasible or they have the same optimal objective.*

Weak duality

Primal $(\mathcal{P})$

$$\begin{array}{llll} & \text{minimize}_x & c^\mathsf{T} x & \\ (\lambda_i \to) & & a_i^\mathsf{T} x \geq b_i, & \forall i \in I_{\text{ge}}, \\ (\lambda_i \to) & & a_i^\mathsf{T} x \leq b_i, & \forall i \in I_{\text{le}}, \\ (\lambda_i \to) & & a_i^\mathsf{T} x = b_i, & \forall i \in I_{\text{eq}}, \\ & & x_j \geq 0, & \forall j \in J_p, \\ & & x_j \leq 0, & \forall j \in J_n, \\ & & x_j \text{ free}, & \forall j \in J_f. \end{array}$$

Dual $(\mathcal{D})$

$$\begin{array}{llll} & \text{maximize}_\lambda & \lambda^\mathsf{T} b & \\ & & \lambda_i \geq 0, & \forall i \in I_{\text{ge}}, \\ & & \lambda_i \leq 0, & \forall i \in I_{\text{le}}, \\ & & \lambda_i \text{ free}, & \forall i \in I_{\text{eq}}, \\ (x_j \to) & & \lambda^\mathsf{T} A_j \leq c_j, & \forall j \in J_p, \\ (x_j \to) & & \lambda^\mathsf{T} A_j \geq c_j, & \forall j \in J_n, \\ (x_j \to) & & \lambda^\mathsf{T} A_j = c_j, & \forall j \in J_f. \end{array}$$

Weak duality

	Primal $(\mathcal{P})$				**Dual** $(\mathcal{D})$	
minimize_{x}	$c^{\mathsf{T}}x$			$\text{maximize}_{\lambda}$	$\lambda^{\mathsf{T}}b$	
$(\lambda_i \rightarrow)$	$a_i^{\mathsf{T}}x \geq b_i,$	$\forall i \in I_{\text{ge}},$			$\lambda_i \geq 0,$	$\forall i \in I_{\text{ge}},$
$(\lambda_i \rightarrow)$	$a_i^{\mathsf{T}}x \leq b_i,$	$\forall i \in I_{\text{le}},$			$\lambda_i \leq 0,$	$\forall i \in I_{\text{le}},$
$(\lambda_i \rightarrow)$	$a_i^{\mathsf{T}}x = b_i,$	$\forall i \in I_{\text{eq}},$			λ_i free,	$\forall i \in I_{\text{eq}},$
	$x_j \geq 0,$	$\forall j \in J_p,$		$(x_j \rightarrow)$	$\lambda^{\mathsf{T}}A_j \leq c_j,$	$\forall j \in J_p,$
	$x_j \leq 0,$	$\forall j \in J_n,$		$(x_j \rightarrow)$	$\lambda^{\mathsf{T}}A_j \geq c_j,$	$\forall j \in J_n,$
	x_j free,	$\forall j \in J_f.$		$(x_j \rightarrow)$	$\lambda^{\mathsf{T}}A_j = c_j,$	$\forall j \in J_f.$

Theorem (Weak duality)

If x is feasible for $(\mathcal{P})$ and λ is feasible for $(\mathcal{D})$, then $\lambda^{\mathsf{T}}b \leq c^{\mathsf{T}}x$.

Proof. Trivially true from our construction – omitted.

Implications of Weak Duality

Corollary

The following results hold:

(c) and (d) provide (sub)optimality certificates, but...

How do we know that the gaps in (c) are not very large?

How do we know that x and λ satisfying (d) even exist?

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The following results hold:

(a) If the optimal objective in $(\mathcal{P})$ *is* $-\infty$*, then* $(\mathcal{D})$ *must be infeasible.*

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(c) If $x \in P$ *and* $\lambda \in D$*, then:*

$$c^\mathsf{T}x - p^\star \leq c^\mathsf{T}x - \lambda^\mathsf{T}b \quad \textbf{and} \quad d^\star - \lambda^\mathsf{T}b \leq c^\mathsf{T}x - \lambda^\mathsf{T}b.$$

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(d) If $x \in P$*,* $\lambda \in D$*, and* $\lambda^\mathsf{T} b = c^\mathsf{T} x$*, then* x **optimal** *for* $(\mathcal{P})$ *and* λ **optimal** *for* $(\mathcal{D})$*.*

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(b) If the optimal objective in $(\mathcal{D})$ is $+\infty$, then $(\mathcal{P})$ must be infeasible.

(c) If $x \in P$ and $\lambda \in D$, then:

$$c^{\mathsf{T}}x - p^{\star} \leq c^{\mathsf{T}}x - \lambda^{\mathsf{T}}b \quad \textbf{and} \quad d^{\star} - \lambda^{\mathsf{T}}b \leq c^{\mathsf{T}}x - \lambda^{\mathsf{T}}b.$$

(d) If $x \in P$, $\lambda \in D$, and $\lambda^{\mathsf{T}}b = c^{\mathsf{T}}x$, then x **optimal** *for $(\mathcal{P})$ and λ* **optimal** *for $(\mathcal{D})$.*

(c) and (d) provide (sub)optimality certificates, but...

How do we know that the gaps in (c) are not very large?

How do we know that x and λ satisfying (d) even exist?