Duality

Lecture 6

October 8, 2025

Quiz

What is the dual of this problem?

$$\begin{aligned} \text{minimize} \quad & x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = 3. \end{aligned}$$

What does this say about the statement: *"In linear optimization, it is possible that the primal problem is infeasible and the dual problem is also infeasible."*?

Recap From Last Time & Today's Plan

Last time...

- **Separating Hyperplane Thm $\Rightarrow$ Farkas Lemma $\Rightarrow$ Strong duality**

Agenda for today:

- Two motivating applications
- Implications of strong duality
- Optimality conditions and primal/dual simplex
- Complementary slackness
- Global sensitivity & Shadow prices as marginal costs
- One more application: network revenue management

Polynomially-Sized CVaR Representation

- Recall homework: ensure CVaR of portfolio payoff exceeds a lower limit
- CVaR was defined as the average over the k-smallest values (for suitable integer k)
- If payoffs in the scenarios are $v_1, v_2, \ldots, v_n$, the key constraint is:

$$\sum_{i=1}^{k} v_{[i]} \geq b, \tag{1}$$

where $v_{[1]} \leq v_{[2]} \leq \cdots \leq v_{[n]}$ is the sorted vector of payoffs.

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- **Claim:**

$$\sum_{i=1}^{k} v_{[i]} = \min_{x \in [0,1]^n} \left\{ \sum_{i=1}^{n} v_i x_i : e^T x = k \right\}. \tag{2}$$

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- **Claim:**

$$\sum_{i=1}^{k} v_{[i]} = \min_{x \in [0,1]^n} \left\{ \sum_{i=1}^{n} v_i x_i \,:\, e^{\mathsf{T}} x = k \right\}. \tag{2}$$

- By strong duality, the optimal value of LP (2) is the same as:

$$\max_{\lambda, t} \left\{ e^{\mathsf{T}} \lambda + k \cdot t \,:\, \lambda + t \cdot e \leq v,\; \lambda \geq 0 \right\}.$$

- So (1) is satisfied if and only: $\exists \lambda, t \,:\, e^{\mathsf{T}} \lambda + k \cdot t \geq b,\; \lambda + t \cdot e \leq v,\; \lambda \geq 0.$

Application in Robust Optimization

- Consider an LP with an uncertain constraint:

$$a^\mathsf{T} x \leq b, \tag{3}$$

  where a satisfies $a \in \mathcal{A}$ and $\mathcal{A}$ is polyhedral

- We seek decisions x that are **robustly feasible**, i.e.,

$$a^\mathsf{T} x \leq b, \forall a \in \mathcal{A} := \{a \in \mathbb{R}^n \,:\, Ca \leq d\} \tag{4}$$

  Infinitely many constraints : "semi-infinite" LP. *Any ideas?*

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- This is a polynomially-sized set of constraints in x, λ

Strong Duality

Consider the following primal-dual pair:

$$(\mathcal{P}) \quad \text{minimize } c^{\mathsf{T}}x \qquad\qquad (\mathcal{D}) \quad \text{maximize } \lambda^{\mathsf{T}}b$$
$$\text{subject to } Ax \geq b \qquad\qquad \text{subject to } \lambda^{\mathsf{T}}A = c^{\mathsf{T}}, \quad \lambda \geq 0.$$

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Theorem (Strong Duality)

If $(\mathcal{P})$ has an optimal solution, so does $(\mathcal{D})$, and their optimal values are equal.

Implications

Strong duality leaves only a few possibilities for a primal-dual pair:

		Dual		
		Finite Optimum	Unbounded	Infeasible
Primal	Finite Optimum	?	?	?
	Unbounded	?	?	?
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Primal	Finite Optimum	Possible	Impossible	Impossible
	Unbounded	Impossible	Impossible	Possible
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Strong Duality and Theorems of Alternative

- Strong duality allows you to **prove** various "theorems of alternative"

Example (Farkas Lemma)

Prove that exactly one of the following is true:
(i) $\exists x \geq 0$ such that $Ax = b$,
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- Set up a (feasibility) problem that mirrors statement (i), and consider its dual.

$$(\mathcal{P}) \quad \max\ 0 \qquad\qquad (\mathcal{D}) \quad \min\ \lambda^\mathsf{T} b$$
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- (i) holds $\Rightarrow p^\star = d^\star = 0 \ \Rightarrow\ \lambda^\mathsf{T} b \geq 0$ for any $\lambda : \lambda^\mathsf{T} A \geq 0$, so (ii) cannot hold.

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- (i) does **not** hold $\Rightarrow d^\star = -\infty \ \Rightarrow \ \exists \lambda : \lambda^\mathsf{T} b < 0$ and $\lambda^\mathsf{T} A \geq 0$, so (ii) holds.

Optimality for Standard-Form LPs

$$(\mathcal{P}) \;\; \min\; c^{\mathsf{T}}x \qquad\qquad (\mathcal{D}) \;\; \max\; \lambda^{\mathsf{T}}b$$
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  - If $B \subseteq \{1, \ldots, n\}$ is a basis, the b.f.s. is: $x = [x_B, 0], \quad x_B = A_B^{-1} b$.
  - Simplex algorithm: feasibility and optimality for $(\mathcal{P})$ are given by:

$$\text{Feasibility-}(\mathcal{P}): \qquad x_B := A_B^{-1} b \geq 0 \tag{6a}$$

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Primal optimality $\Leftrightarrow$ Dual feasibility

Simplex terminates when finding a dual-feasible solution!

Solve $(\mathcal{P})$ or $(\mathcal{D})$?

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Primal simplex

- maintain a **basic feasible solution**
- basis $B \subset \{1, \ldots, n\}$
- stopping criterion: dual feasibility

Dual simplex

- maintain a dual feasible solution
- stopping criterion: primal feasibility
- different from primal simplex: works with an LP with inequalities

- How to choose $(\mathcal{P})$ or $(\mathcal{D})$?
- Suppose we have $x^\star$, $\lambda^\star$ and must now solve a **larger** problem, i.e., with extra decisions or extra constraints.
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- *Any preference between primal and dual simplex?*
  - With extra decisions $x_e \Rightarrow$ **primal simplex** initialized with $[x^\star, x_e = 0]$.
  - With extra constraints $A_e x = b_e \Rightarrow$ **dual simplex** initialized with $[\lambda^\star, p_e = 0]$.
- Modern solvers include primal and dual simplex and allow concurrent runs

Optimality Conditions and Complementary Slackness

Primal-Dual Pair of Problems

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Consider $x \in P, \lambda \in D$ (each feasible). How to check if they are **optimal**?

Optimality Conditions and Complementary Slackness

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Theorem (Complementary Slackness)

$x \in P$ and $\lambda \in D$ are **optimal** *solutions for $(\mathcal{P})$ and $(\mathcal{D})$, respectively,* **if and only if***:*

$$\lambda_i(a_i^{\mathsf{T}}x - b_i) = 0,\ i = 1, \dots, m$$
$$(\lambda^{\mathsf{T}}A_j - c_j)x_j = 0,\ j = 1, \dots, n.$$

- Follows from primal/dual feasibility and $c^{\mathsf{T}}x = b^{\mathsf{T}}\lambda$

Optimality Conditions and Complementary Slackness

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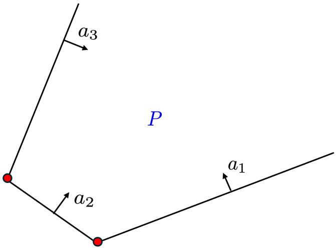
- Follows from primal/dual feasibility and $c^\mathsf{T} x = b^\mathsf{T} \lambda$
- Interesting insight: **non-binding constraint** $\Rightarrow$ dual variable is **zero**

Representation of Polyhedra

Important consequence of duality: alternative representation of all polyhedra

Definition

Consider a nonempty polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$. Then:



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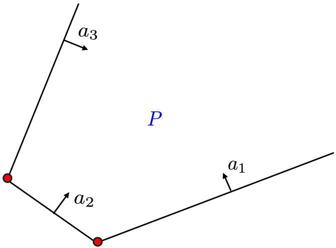
1. $\mathcal{C} := \{d \in \mathbb{R}^n : Ad \geq 0\}$ is called the **recession cone** of P.
2. Any $d \in \mathcal{C}$ with $d \neq 0$ is called a **ray** of P.



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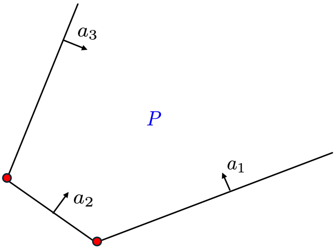

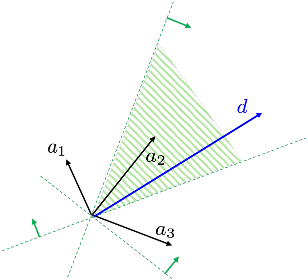

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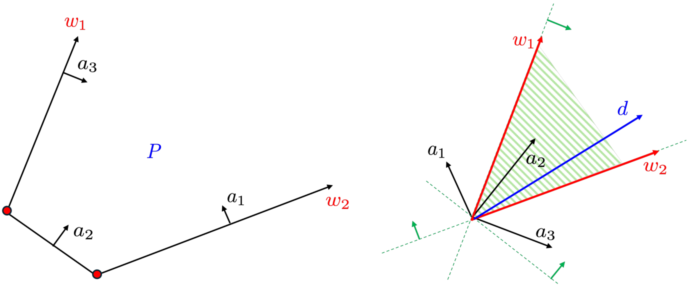
3. Any ray d that satisfies $a_i^\mathsf{T} d = 0$ for $n-1$ linearly independent a_i is called an **extreme ray** of P.



Representation of Polyhedra

Theorem (Resolution Theorem)

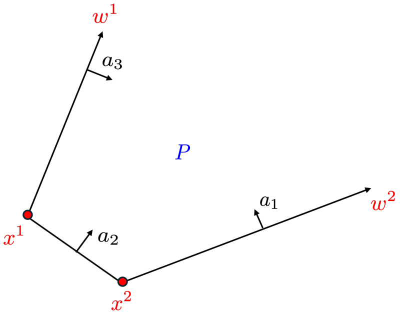
Let $P = \{x \in \mathbb{R}^n : Ax \geq b\}$ *be a non-empty polyhedron,* $x^1, x^2, \ldots, x^k$ *be its* **extreme points**, *and* $w^1, w^2, \ldots, w^r$ *be its* **extreme rays**. *Then,*

Representation of Polyhedra

Theorem (Resolution Theorem)

Let $P = \{x \in \mathbb{R}^n \ : \ Ax \geq b\}$ be a non-empty polyhedron, $x^1, x^2, \ldots, x^k$ be its **extreme points**, *and $w^1, w^2, \ldots, w^r$ be its* **extreme rays**. *Then,*

$$P = \text{conv}\big(\{x^1, \ldots, x^k\}\big) + \text{cone}\big(\{w^1, \ldots, w^r\}\big)$$
$$= \left\{ \sum_{i=1}^{k} \mu_i x^i + \sum_{j=1}^{r} \theta_j w^j \ : \ \mu \geq 0, \ e^\mathsf{T}\mu = 1, \ \theta \geq 0 \right\}.$$



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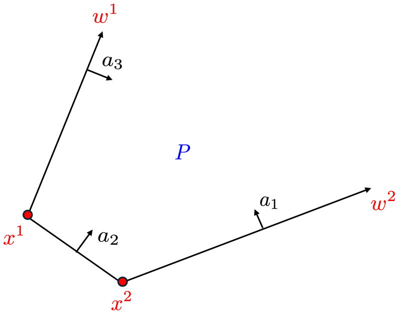


Note: It is **not** "easy" (i.e., poly-time) to switch between these representations

Dual Variables As Marginal Costs

$$(\mathcal{P})\ \min\ c^{\mathsf{T}}x \qquad\qquad (\mathcal{D})\ \max\ \lambda^{\mathsf{T}}b$$

$$Ax = b, \quad x \geq 0 \qquad\qquad \lambda^{\mathsf{T}}A \leq c^{\mathsf{T}}$$

- Solved the LP and obtained $x^\star$ and $\lambda^\star$
- Want to show that $\lambda^\star$ is the **gradient of the optimal cost with respect to** b "almost everywhere"
- Related to **sensitivity analysis**
  How do the optimal value and solution depend on problem data A, b, c*?*

Sensitivity: A Simple Example

Maximize Y

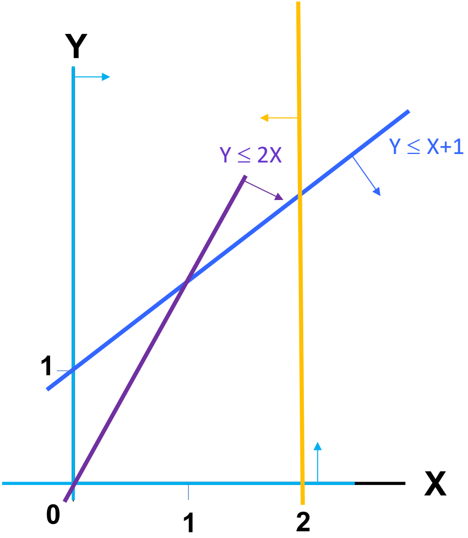
Subject to:

$Y \leq 2X$

$Y \leq X+1$

$X \geq 0, Y \geq 0$

$X \leq 2$



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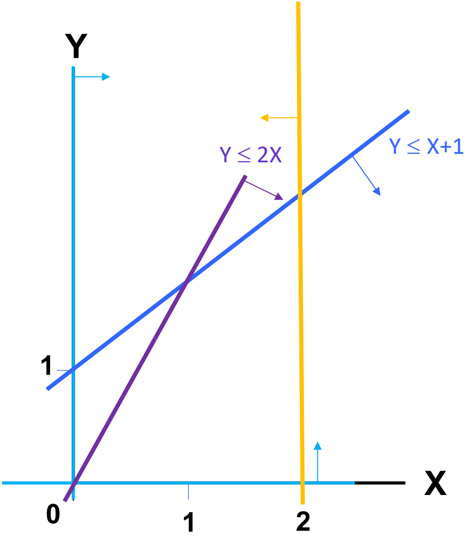

For the last constraint $X \leq a$, what is the shadow price i.e., rate of change in the optimal value when we change the constraint r.h.s. a?

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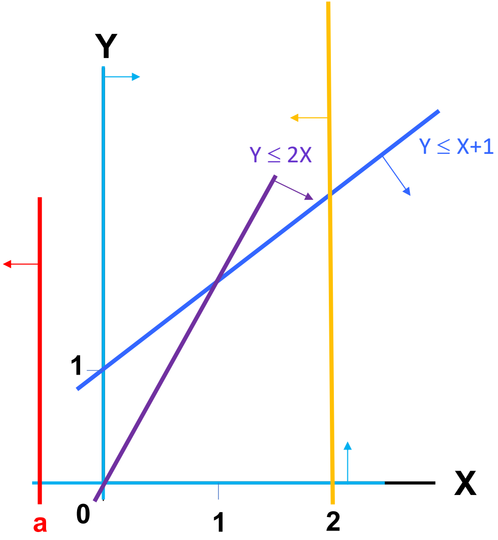

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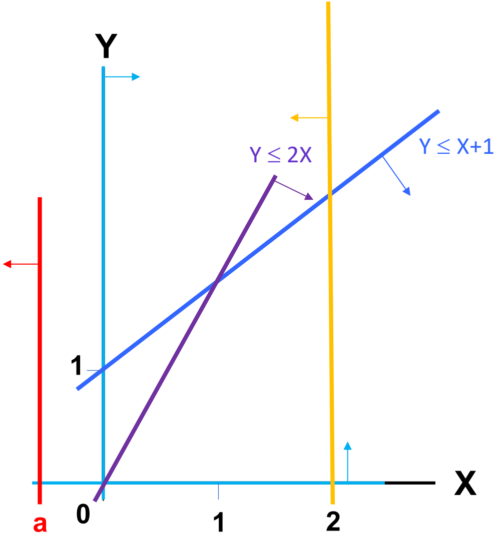
If a < 0 :
- **Infeasible!**



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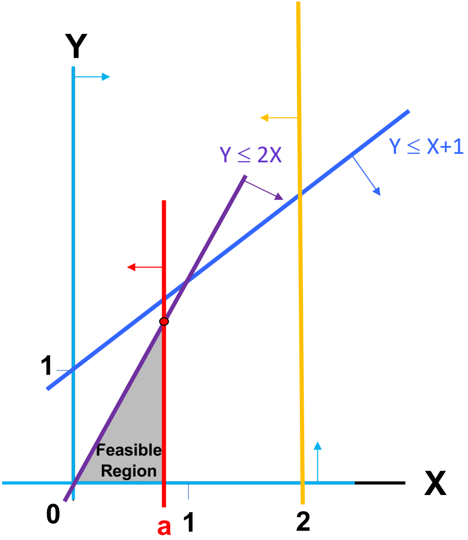

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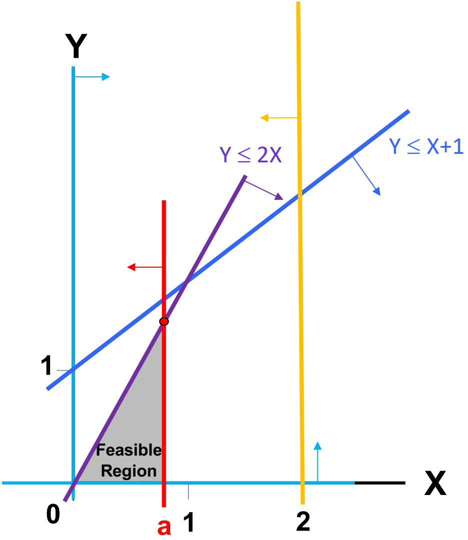
If $0 < a < 1$:

- Shadow price = 2



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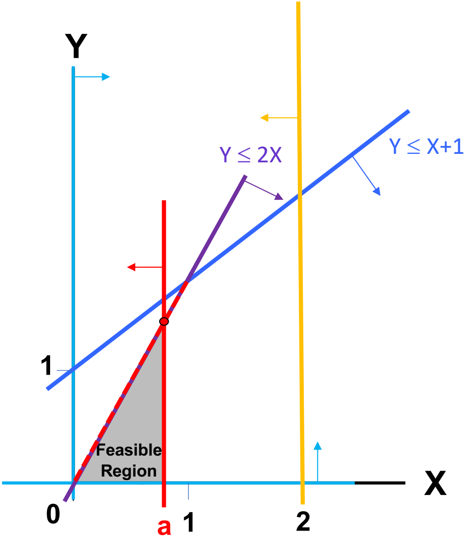

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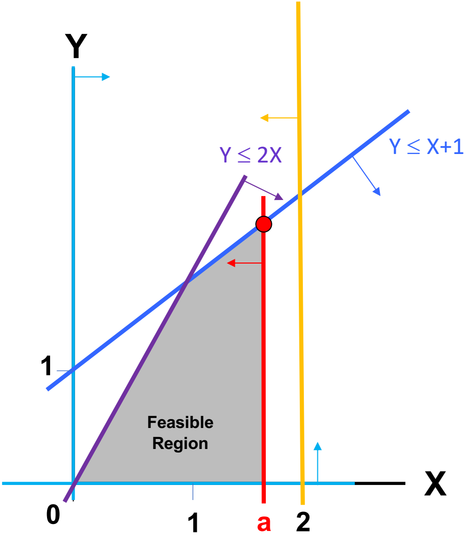

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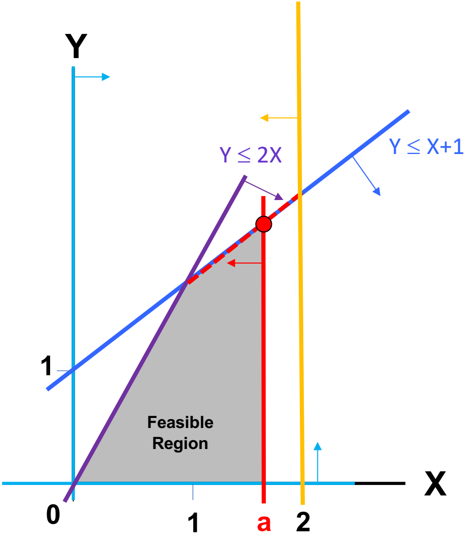

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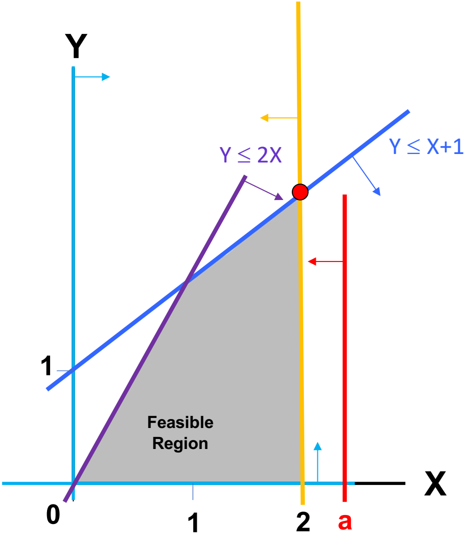
If a > 2 :



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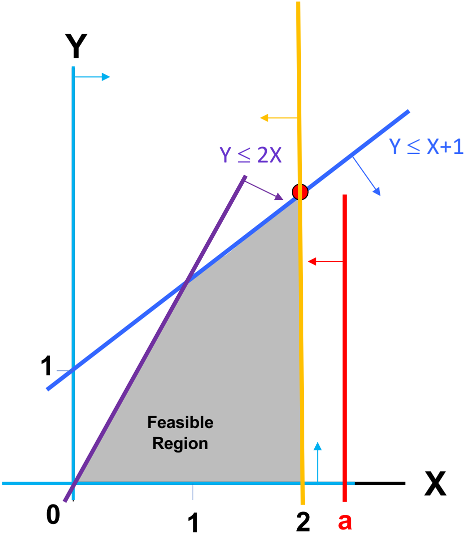

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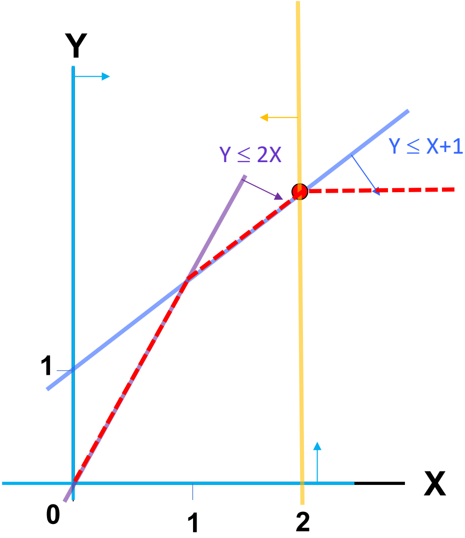
Note how the objective depends on *a* overall



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- What to show that the **optimal value** (when finite) **as a function of** b is
- What to show that the **optimal value** (when finite) **as a function of** c is

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- What to show that the **optimal value** (when finite) **as a function of** b is piecewise linear and **convex**
- What to show that the **optimal value** (when finite) **as a function of** c is piecewise linear and **concave**

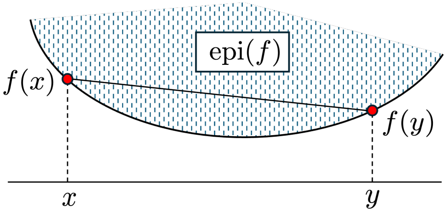
Convex and Concave Functions

Definition

$f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is **convex** if X is a convex set and

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y), \quad \forall x, y \in X \text{ and } \lambda \in [0,1]. \tag{8}$$

A function is **concave** if $-f$ is convex.



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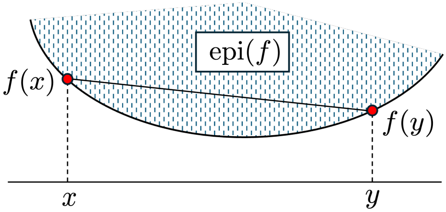

Equivalent definition in terms of **epigraph**:

$$\text{epi}(f) = \{(x,t) \in X \times \mathbb{R} \, : \, t \ge f(x)\} \tag{9}$$

f is convex if and only if epi(f) is a convex set.

Global Dependency On b

- Let $P(b) := \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ denote the feasible set of the primal
- Let $S := \{b \in \mathbb{R}^m : P(b) \neq \emptyset\}$: right-hand-side values that yield a feasible primal
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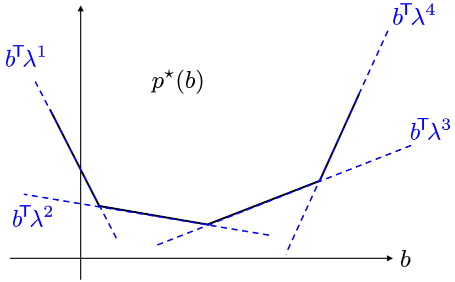
- If $\lambda^1, \lambda^2, \ldots, \lambda^r$ are the extreme points of D, then: $p^\star(b) = \max_{i=1,\ldots,r} b^\mathsf{T}\lambda^i$, $\forall\, b \in S$



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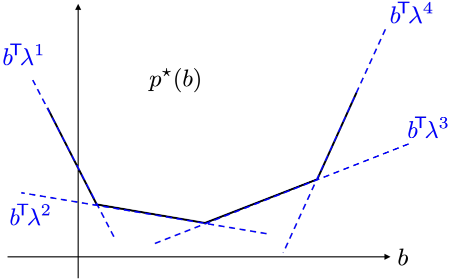

How to prove $p^\star(b)$ convex?

$$\text{epi}(p^\star) = \cap_{i=1,\ldots,r}\text{epi}(b^\mathsf{T}\lambda^i)$$

is the intersection of convex sets, so it is convex.

Global Dependency On b - Implications

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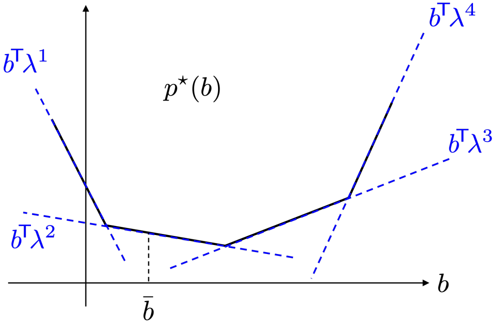

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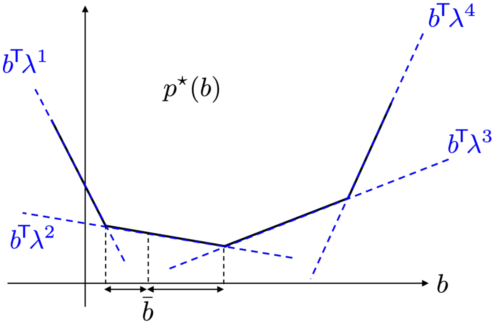

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- $\lambda_i^\star$ acts as a **marginal cost** or **shadow price** for the i-th constraint r.h.s. b_i
- λ_i allows estimating **exact change in $p^\star$ in a range around $\bar{b}_i$**

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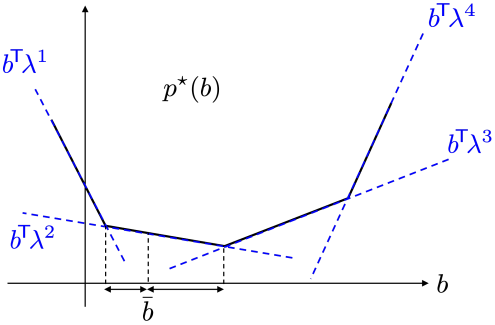

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- λ_i allows estimating **exact change in $p^\star$ in a range around $\bar{b}_i$**
- Modern solvers give **direct access to $\lambda_i^\star$ and the range**

  Gurobipy: for constraint **c**, the attribute **c.Pi** is $\lambda_i^\star$ and the range is from **c.SARHSLow** to **c.SARHSUp**

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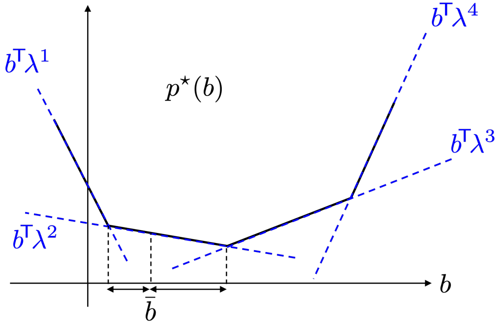

- At b where $p^\star$ is not differentiable, several λ^i are optimal
- All such λ^i are valid **subgradients** of $p^\star$

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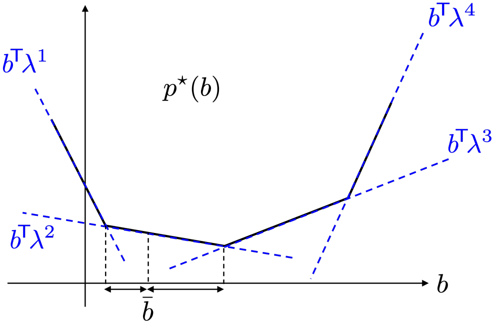

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Definition (Subgradient.)

$f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ convex function. A vector $g \in \mathbb{R}^n$ is a **subgradient** of f at $\bar{x} \in S$ if

$$f(x) \geq f(\bar{x}) + g^\top (x - \bar{x}), \quad \forall x \in S.$$

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- Let $d^\star(c)$ denote optimal value as function of c; assume $d^\star(c) > -\infty$
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- $d^\star(c)$ is a **concave** function of c on the set $T := \{c : d^\star(c) > -\infty\}$
- If for some c the LP has a **unique** optimal solution $x^\star$, then $d^\star$ is linear in the vicinity of c and its gradient is $x^\star$.

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- The optimal primal solution $x^\star$ **is a shadow price for the dual constraints**
- $x^\star$ remains optimal for a range of change in each objective coefficient c_j
- Modern solvers also allow obtaining the range directly

  Gurobipy: attributes **SAObjLow** and **SAObjUp** for each decision variable

Signs of Dual Variables Revisited

- There is a direct connection between:
  - **the optimization problem** (max/min)
  - the **constraint type** ($\leq$, $\geq$)
  - **the signs of the shadow prices**
- Given two of these, can figure out the third one!
- *What is the sign of the shadow price for a ...*

  $\leq$ constraint in a **minimization** problem ?
  $\geq$ constraint in a **minimization** problem ?
  $\leq$ constraint in a **maximization** problem ?
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- *What is the dependency of the optimal objective on the r.h.s. of a ...*

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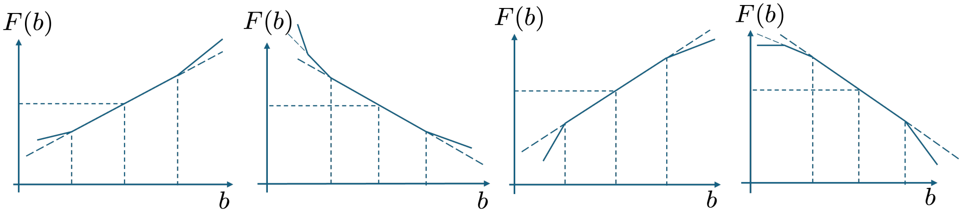
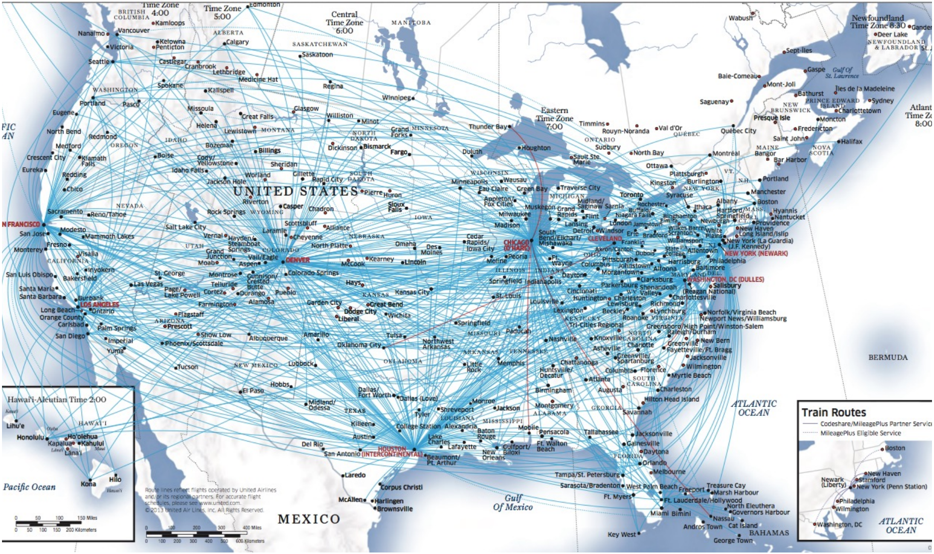
$\min, \geq b$	$\min, \leq b$	$\max, \leq b$	$\max, \geq b$
dual ≥ 0	dual ≤ 0	dual ≥ 0	dual ≤ 0
$F(b)$ convex	$F(b)$ convex	$F(b)$ concave	$F(b)$ concave



Real-World Hub and Spoke Airline Network



Source: www.united.com

Airline Revenue Management (RM)

- **Strategic RM**
  - Determine several price points for various itineraries
  - “Product” or “itinerary”: origin, destination, day, time, various restrictions, …
    - E.g., JFK – ORD – SFO, 10:30am on Oct 12, 2024, Economy class Y fare
  - Typically done by (or in conjunction with) marketing department
    - Market segmentation; competition

- **Tactical RM** (“yield management”) decides **booking limits**
  - A *booking limit* determines how many seats to reserve for each “product”
  - RM not based on setting prices, but rather changing availability of fare classes
  - Legacy due to original IT systems used (e.g., SABRE)

Airline RM

Hub: Chicago ORD

Two planes

Westbound flights for some day in the future

SFO

BOS

ORD

LAX

JFK

Airline RM

Flight segments (legs)

SFO

BOS

ORD

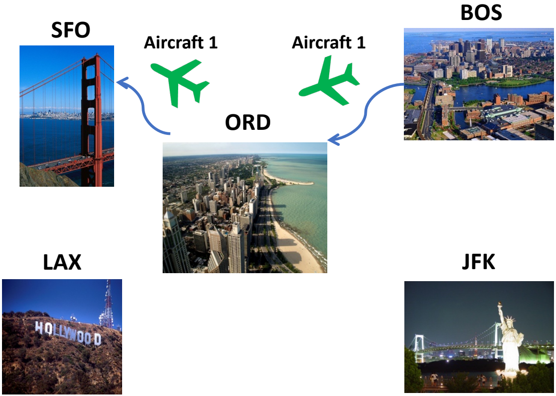
LAX

JFK

Airline RM

Flight segments (legs)

- **Aircraft 1:**
  - BOS-ORD in the morning
  - ORD-SFO in the afternoon



Airline RM

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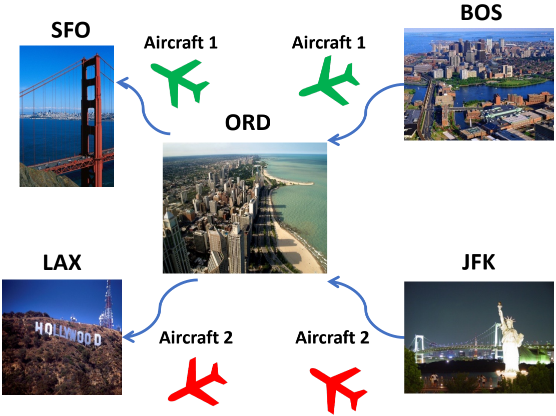

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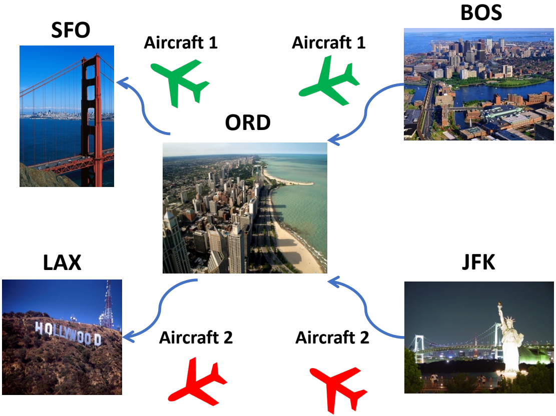

Itineraries

Origin-Destination	Q_Fare	Y_Fare
BOS_ORD	$200	$220
BOS_SFO	$320	$420
BOS_LAX	$400	$490
JFK_ORD	$250	$290
JFK_SFO	$410	$540
JFK_LAX	$450	$550
ORD_SFO	$210	$230
ORD_LAX	$260	$300

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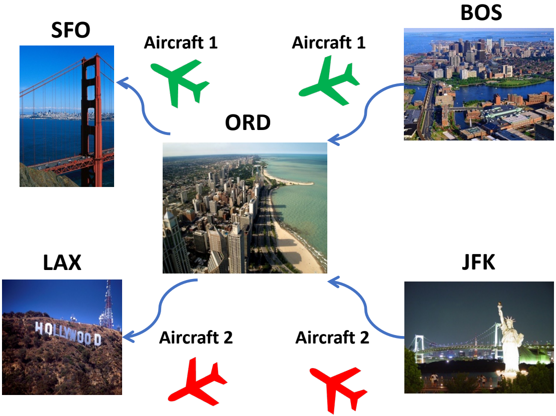

Itineraries

Origin-Destination	Q_Fare	Y_Fare	Q_Demand	Y_Demand
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JFK_ORD	$250	$290	24	16
JFK_SFO	$410	$540	65	50
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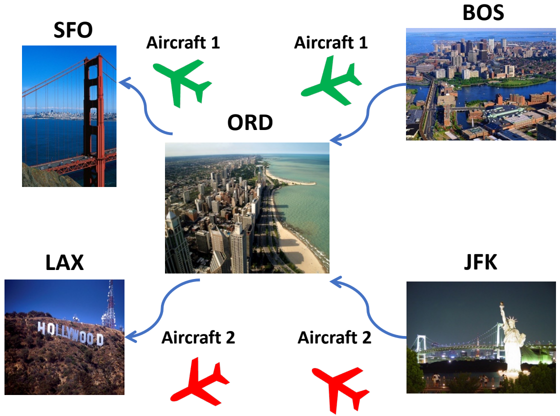

Resources needed

	BOS_ORD	BOS_SFO	BOS_LAX	JFK_ORD	JFK_SFO	JFK_LAX	ORD_SFO	ORD_LAX
Flight leg								
BOS_ORD_Leg	1	1	1	0	0	0	0	0
JFK_ORD_Leg	0	0	0	1	1	1	0	0
ORD_SFO_Leg	0	1	0	0	1	0	1	0
ORD_LAX_Leg	0	0	1	0	0	1	0	1

Network Revenue Management

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Resource matrix A :

	Itinerary 1	Itinerary 2	...	Itinerary $\|I\|$
Flight leg 1	1	0	...	1
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⋮	⋮	⋮	⋮	⋮
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- Goal: decide how many itineraries of each type to sell to maximize revenue

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- **Bid-price heuristic** in network revenue management
- Broader principle of how to price "products" through resource usage/cost