Lecture 8 : Duality in Convex Optimization

October 15, 2025

Today's Agenda: Convex Duality

Primal Problem

$$(\mathcal{P})\ \text{minimize}_x \quad f_0(x) \qquad f_i(x) \leq 0, \quad i = 1, \dots, m \qquad x \in X. \tag{1}$$

- Convex domain $X \subseteq \mathbb{R}^n$
- Every function $f_i : X \subseteq \mathbb{R}^n \to \mathbb{R}$ (real-valued), **convex**
- Equality constraints $Ax = b$ can be included in X

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- Many developments deal with the "interior" of X

Definition : Interior

The **interior** of a set X is the set of all points $x \in X$ so that:

$$\exists\, r > 0 \ : \ B(x, r) := \{y : \|y - x\| \le r\} \subseteq X$$

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What is the interior of a set X that is not full-dimensional?

Relative Interior

- **Recall**: **Affine hull** of X is $\text{aff}(X) := \left\{\theta_1 x_1 + \cdots + \theta_k x_k \ : \ x_i \in X, \ \sum_{i=1}^k \theta_i = 1\right\}$

Relative Interior

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Definition Relative Interior

The **relative interior** of a set X is:

$$\mathrm{rel\,int}(X) := \left\{x \in X \ : \ \exists\, r > 0 \text{ so that } B(x, r) \cap \mathrm{aff}(X) \subseteq X\right\}. \tag{2}$$

What is the relative interior of the following sets?

- $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \in [0, 1]^2\}$
- $\{(x, y) \in \mathbb{R}^2 \mid x + y = 1, x \geq 0, y \geq 0\}$
- $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

Convex Duality

Primal Problem

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- Assume $\operatorname{rel\,int}(X) \neq \emptyset$
- Assume that $(\mathcal{P})$ has an optimal solution $x^\star$, optimal value $p^\star = f_0(x^\star)$

- **Core questions:**
  1. For x feasible for $(\mathcal{P})$, how to **quantify the optimality gap** $f_0(x) - p^\star$?
  2. How to certify that $x^\star$ is **optimal** in $(\mathcal{P})$?

Convex Duality

Primal Problem

$$(\mathcal{P}) \text{ minimize}_x \quad f_0(x) \quad f_i(x) \leq 0, \quad i = 1, \dots, m \quad x \in X.$$

- To construct **lower bounds** for $(\mathcal{P})$, define the **Lagrangian function**: for $\lambda \geq 0$,

$$\mathcal{L}(x, \lambda) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x)$$

Convex Duality

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- By construction, $\mathcal{L}(x, \lambda) \le f_0(x)$ for any x feasible in $(\mathcal{P})$

Convex Duality

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- For a lower bound on $p^\star$, minimize $\mathcal{L}(x, \lambda)$ over $x \in X$ to get:

$$g(\lambda) := \inf_{x \in X} \mathcal{L}(x, \lambda).$$

Dual Problem

$$(\mathcal{D}) \quad \sup_{\lambda \geq 0} g(\lambda).$$

Q: Is the dual $(\mathcal{D})$ a convex optimization problem?

Convex Duality

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Dual Problem

$$(\mathcal{D}) \quad \sup_{\lambda \geq 0} g(\lambda).$$

Q: Is the dual $(\mathcal{D})$ a convex optimization problem? Yes, even if $(\mathcal{P})$ isn't!

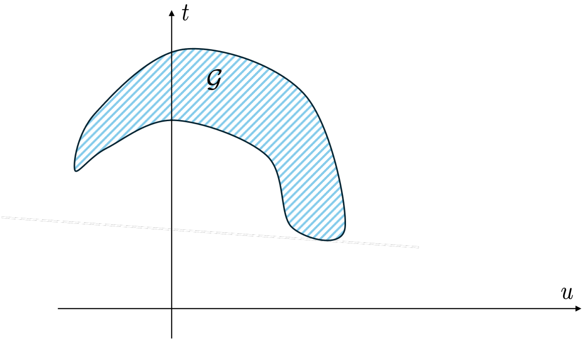
Geometric Interpretation

Primal-Dual Pair

$$(\mathcal{P})\ p^\star := \inf_{x \in X} f_0(x) \qquad\qquad (\mathcal{D}) \quad d^\star := \sup_{\lambda \geq 0} g(\lambda)$$

$$f_i(x) \leq 0,\ i = 1, \ldots, m$$

- Suppose $X = \mathbb{R}^n$ and $(\mathcal{P})$ has just one inequality constraint, i.e., $m = 1$
- Let $\mathcal{G} := \{(u, t) \in \mathbb{R}^2 : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$



What do feasible points in $(\mathcal{P})$ correspond to? Where is $p^\star$?

How to express the Lagrangian $\mathcal{L}(x, \lambda)$ using the t, u variables?

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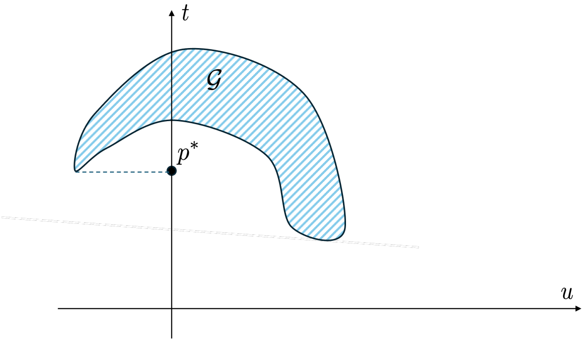

$\mathcal{L}(x, \lambda)$ is the same as $t + \lambda \cdot u$.

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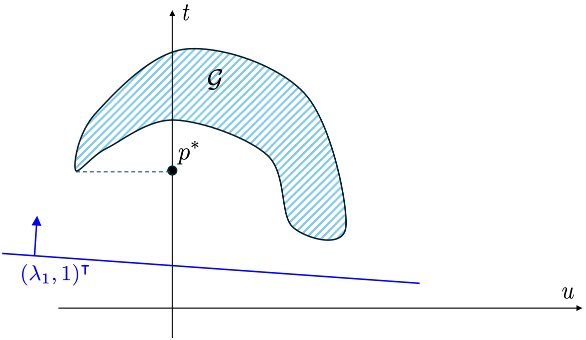

For $\lambda \geq 0$, we have $g(\lambda) = \inf_{x\in X}\big(f_0(x) + \lambda f_1(x)\big) = \inf_{(u,t)\in\mathcal{G}}(t + \lambda \cdot u)$

What is the value of $g(\lambda_1)$ in this figure?

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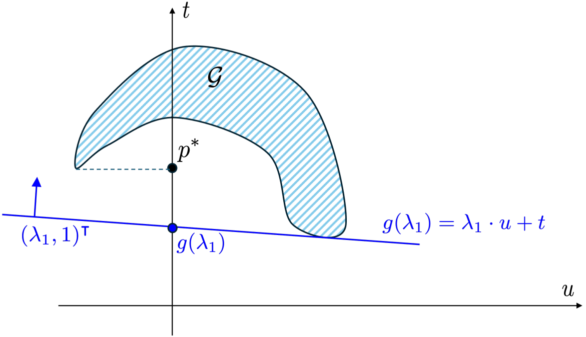

The optimal pairs (u, t) yield a **supporting hyperplane** for $\mathcal{G}$

Intersection with $u = 0$ is value of $g(\lambda_1)$

Geometric Interpretation

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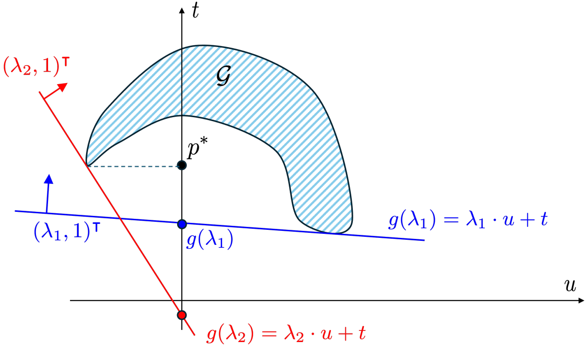

What is the value of $\max_{\lambda \geq 0} g(\lambda)$**?**

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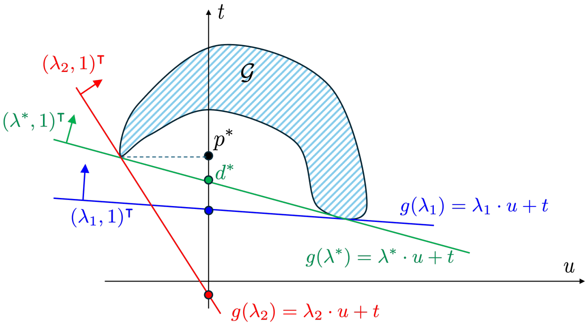

Here, strong duality does not hold: $d^\star < p^\star$. But the set $\mathcal{G}$ is not convex!

Strong Duality in Convex Optimization?

Strong Duality in Convex Optimization?

Non-zero duality gap

Let $X = \{(x, y) \mid y \geq 1\}$ and consider the problem:

$$\underset{(x,y)\in X}{\text{minimize}} \; e^{-x}$$

$$x^2/y \leq 0.$$

- **Is this a convex optimization problem?**
- **What are $p^\star$, $\mathcal{L}$, g, $d^\star$?**
- **Does $p^\star = d^\star$ hold for any primal convex optimization problem if $p^\star$ finite?**

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$$g(\lambda) = \inf_{x,y\geq 1}\left(e^{-x} + \lambda \frac{x^2}{y}\right) = 0 \text{ for any } \lambda \geq 0.$$

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- We can write the dual problem as $d^\star = \max_{\lambda\geq 0} 0$, with optimal value $d^\star = 0$
- We have a duality gap: $p^\star - d^\star = 1$
- Primal and dual both have finite optimal value, but a gap exists!
- Examples also exist where $(\mathcal{D})$ does not achieve its optimal value... (notes)

Conditions Leading to Strong Duality

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Slater Condition

The functions $f_1, \ldots, f_m : X \subseteq \mathbb{R}^n \to \mathbb{R}$ satisfy **the Slater condition on** X if there exists $x \in \operatorname{rel\,int}(X)$ such that

$$f_j(x) < 0, \quad j = 1, \ldots, m.$$

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- A point x that is **strictly feasible**
- If **all** $f_i(x)$ are **affine**, we do not need this (i.e., feasibility is enough)
- If **some** f_i are affine, we only require $f_i(x) < 0$ for the **non-linear** f_i

Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \ldots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

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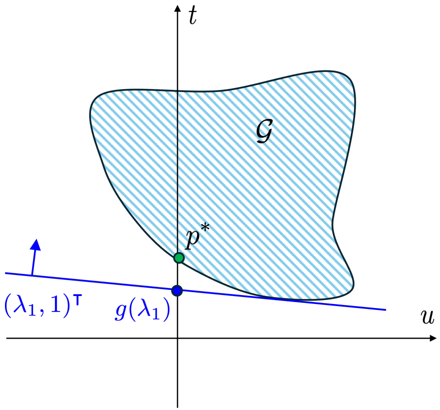

Geometric intuition for proof:

- Recall $\mathcal{G} := \{(u, t) \in \mathbb{R}^{m+1} : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u_i = f_i(x), i = 1, \ldots, m\}$

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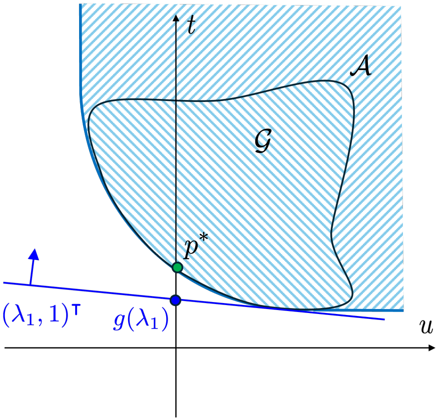

- Recall $\mathcal{G} := \{(u,t) \in \mathbb{R}^{m+1} : \exists x \in \mathbb{R}^n,\ t = f_0(x),\ u = f_1(x)\}$ (above, $m = 1$)
- Same $p^\star$ if we replace $\mathcal{G}$ with $\mathcal{A} = \{(u,t) \in \mathbb{R}^{m+1} : \exists x \in \mathbb{R}^n, t \geq f_0(x), u_i \geq f_i(x), \forall i\}$

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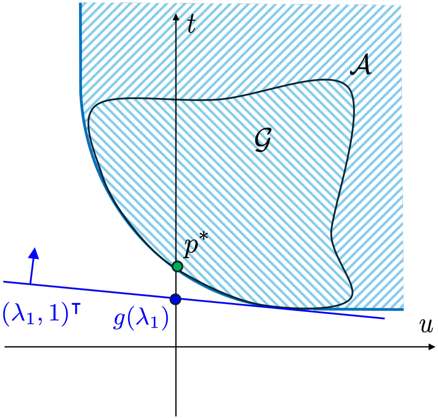

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- **Is $\mathcal{A}$ a convex set?**

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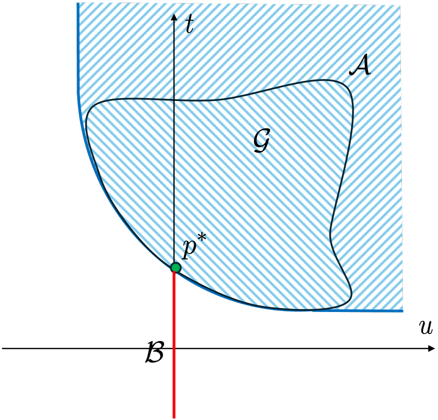

- Define $\mathcal{B} := \{(0, t) \in \mathbb{R}^m \times \mathbb{R} \mid t < p^\star\}$
- **Claim.** $\mathcal{A} \cap \mathcal{B} = \emptyset$

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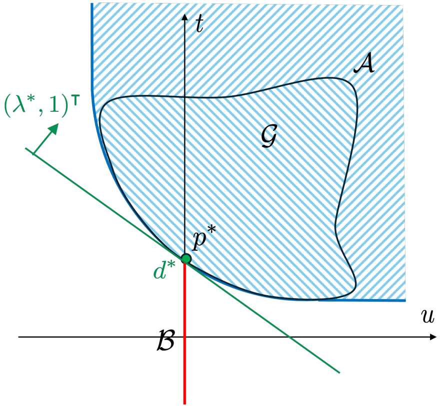

- The Separating Hyperplane Theorem will give us the optimal $\lambda^\star$ and $p^\star = d^\star$

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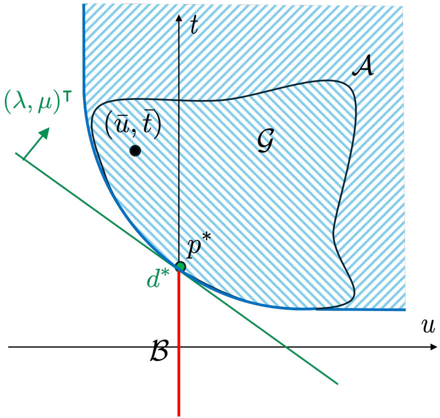

- The Slater point will guarantee that the hyperplane is not vertical

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- Define the **convex** set

$$\mathcal{A} = \big\{(u, t) \in \mathbb{R}^m \times \mathbb{R} : \exists\, x \in X, \quad t \geq f_0(x), u_i \geq f_i(x),\ i = 1, \ldots, m\big\}.$$

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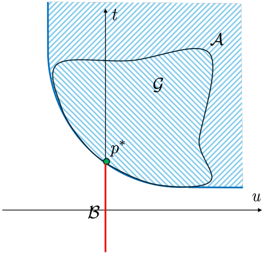

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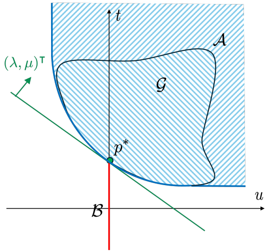
- Define the **convex** set $\mathcal{B} = \{(0,t) \in \mathbb{R}^m \times \mathbb{R} \mid t < p^\star\}$.
- $\mathcal{A} \cap \mathcal{B} = \emptyset$.
- (Non-strict) Separating Hyperplane Theorem:

$$\exists\, (\lambda, \mu) \in \mathbb{R}^{m+1},\ \ b \in \mathbb{R} : \begin{cases} (1) \quad (\lambda, \mu) \neq 0, \\ (2) \quad \lambda^\mathsf{T} u + \mu t \geq b,\ \forall (u,t) \in A \\ (3) \quad \lambda^\mathsf{T} u + \mu t \leq b,\ \forall (u,t) \in B. \end{cases}$$



Strong Duality in Convex Optimization

Theorem (Strong Duality in Convex Optimization)

Let $X \subset \mathbb{R}^n$ be convex and $f_0, f_1, \dots, f_m : X \to \mathbb{R}$ convex functions on X satisfying the Slater condition on X. Then, $p^\star = d^\star$ and the dual attains its optimal value.

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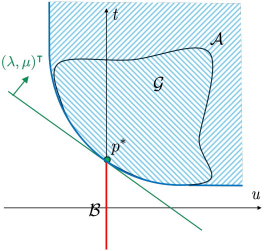

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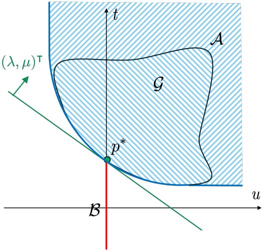

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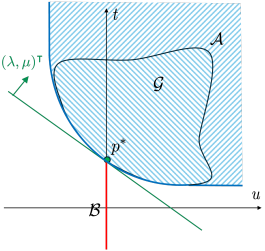

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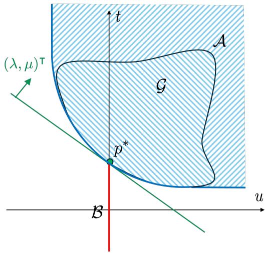

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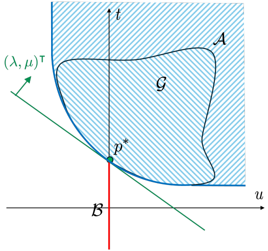

- **Case 1.** $\mu > 0$ (non-vertical hyper-plane)
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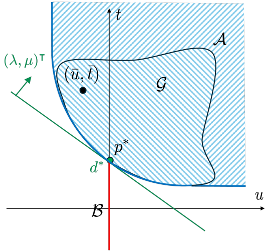

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- Weak duality: $g(\lambda/\mu) \leq p^\star$, so $g(\lambda/\mu) = p^\star$.
- Strong duality holds: $p^\star = d^\star$.

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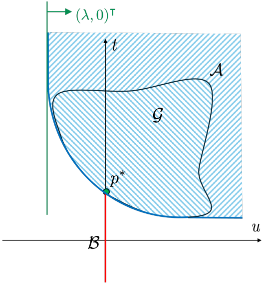

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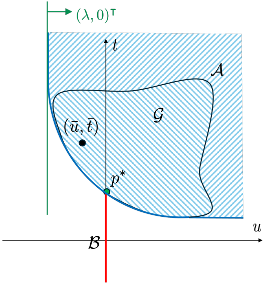

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- $\bar{x}$ satisfies Slater condition $\Rightarrow f_i(\bar{x}) < 0$ for $i = 1, \ldots, m$
- This together with $\lambda \geq 0$ implies that $\lambda = 0$
- Contradicts that $(\lambda, \mu) \neq 0$.

Explicit Equality Constraints

- In applications, useful to make the **equality constraints explicit**:

$$\begin{aligned} \text{minimize}_{x \in X} \; & f_0(x) \\ \text{subject to} \; & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & Ax = b. \end{aligned}$$

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$$\begin{aligned} \text{maximize}_{\lambda, \nu} \; & g(\lambda, \nu) \\ \text{subject to} \; & \lambda \geq 0. \end{aligned}$$

  No sign constraints on ν!

Minimum Euclidean Distance Problem

- Given $y \in \mathbb{R}^n$ and affine set $\{z : Az = \tilde{b}\}$
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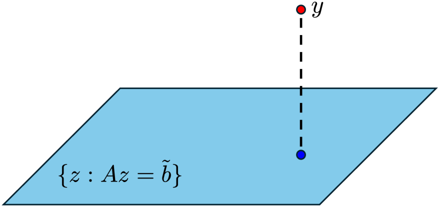
$$\min_z \left\{ \|z - y\|_2^2 \ : \ Az = \tilde{b} \right\}$$



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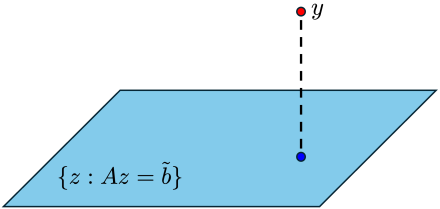

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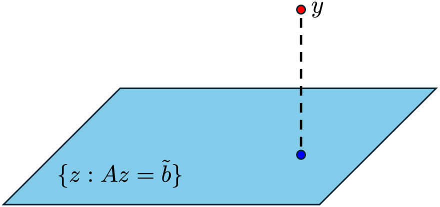

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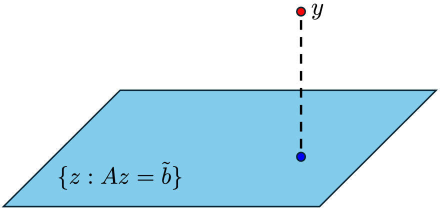

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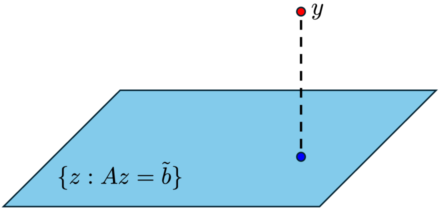

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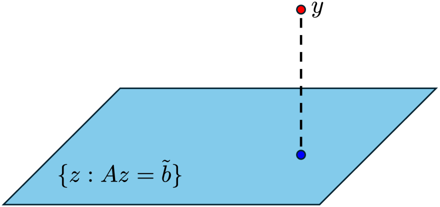

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- AA^T is invertible, so $\nu^\star = -2(AA^T)^{-1} b$, $p^\star = d^\star = g(\nu^\star) = b^T (AA^T)^{-1} b$
- $x^\star = -\frac{1}{2} A^T \nu^\star = A^T (AA^T)^{-1} b$

Quadratic Programs - Preliminaries

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For $Q = Q^\mathsf{T}$, consider the following unconstrained problem:

$$\min\ f(x) := \frac{1}{2}x^\mathsf{T}Qx + q^\mathsf{T}x$$

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$$\nabla_x f(x) = 0 \quad \Leftrightarrow \quad Qx = -q$$

$$p^\star = \begin{cases} -\frac{1}{2}q^\mathsf{T}Q^\dagger q & \text{if } Q \succeq 0 \text{ and } q \in \mathcal{R}(Q) \\ -\infty & \text{otherwise.} \end{cases}$$

- $Q^\dagger$ is the (Moore-Penrose) pseudo-inverse of Q
- For A with singular value decomposition $A = U\Sigma V^\mathsf{T}$, $A^\dagger := V\Sigma^{-1}U^\mathsf{T}$
- Equals $(A^\mathsf{T}A)^{-1}A^\mathsf{T}$ if $\text{rank}(A) = n$ and $A^\mathsf{T}(AA^\mathsf{T})^{-1}$ if $\text{rank}(A) = m$