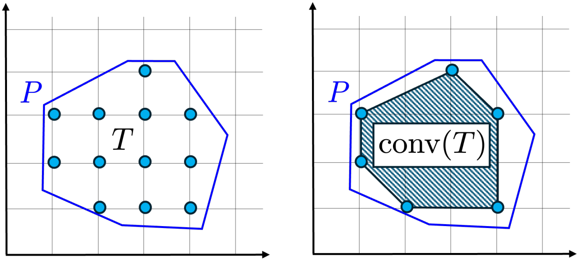
Lecture 17

November 19, 2024

Recall from Monday: Strength of IP Formulation



- Consider an IP with bounded feasible set
  - T : all feasible points to the IP
  - P : feasible set for LP relaxation to IP
  - $\mathrm{conv}(T)$: the convex hull of T (a polyhedral set)
  - Always have: $T \subseteq \mathrm{conv}(T) \subseteq P$.

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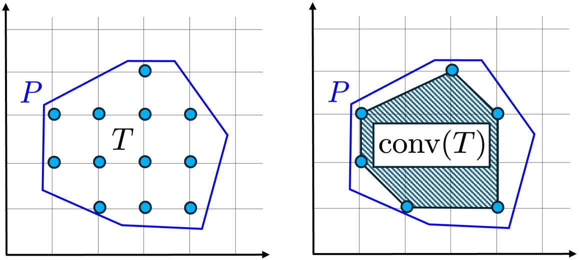

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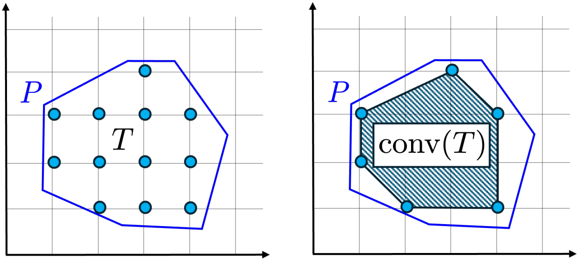

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  - T : all feasible points to the IP
  - P : feasible set for LP relaxation to IP
  - $\mathrm{conv}(T)$: the convex hull of T (a polyhedral set)
  - Always have: $T \subseteq \mathrm{conv}(T) \subseteq P$.
- **Ideal IP formulation:** $P = \mathrm{conv}(T)$

1. Discuss a few **ideal formulations** : $P = \mathrm{conv}(T)$
2. Discuss how to **improve** formulations by adding **cuts**
3. Discuss **algorithms/solution approaches**

(Total) Unimodularity : Ideal Formulations

Definition

1. $A \in \mathbb{Z}^{m \times n}$ of full row rank is **unimodular** if $\det(A_B) \in \{1, -1\}$ for every basis B.

2. $A \in \{-1, 0, 1\}^{m \times n}$ is **totally unimodular** if the determinant of each square submatrix of A is 0, 1, or -1.

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Theorem

1. $A \in \mathbb{Z}^{m \times n}$ **unimodular if and only if** $P(b) = \{x \in \mathbb{R}^n_+ \mid Ax = b\}$ **is integral** *for all* $b \in \mathbb{Z}^m$ *with* $P(b) \neq \emptyset$.

2. A **is totally unimodular if and only if** $P(b) = \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ **is integral** *for all* $b \in \mathbb{Z}^m$ *with* $P(b) \neq \emptyset$.

Checking for Total Unimodularity

Proposition (Refreshed; **sufficient**, but **not necessary**.)

A matrix $A \in \{0, 1, -1\}^{m \times n}$ is totally unimodular if any of the following holds:

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3. *$[A \;\; -A \;\; I \;\; -I]$ is totally unimodular.*
4. *Every subset R of rows of A can be partitioned into R_1 and R_2 so that $\sum_{i \in R_1} a_i - \sum_{i \in R_2} a_i \in \{0, +1, -1\}$. (By 1, a similar result holds for **columns** of A.)*

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5. *Each column of A contains at most two nonzero elements and the rows of A can be partitioned into R_1 and R_2 so that the two nonzero entries in a column are in the same R_i if they have different signs and are in different R_i if they have the same sign.*

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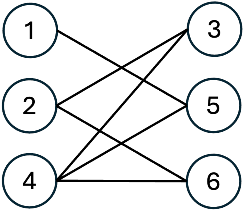
7. *A has the* **consecutive ones** *property: for every column j, $a_{sj} = a_{tj} = 1$ implies $a_{ij} = 1$ for $s \leq i \leq t$.*

Examples of TU Matrices #1

- $G = (\mathcal{N}, \mathcal{E})$ **undirected graph**

- $A \in \{0,1\}^{|\mathcal{N}| \times |\mathcal{E}|}$ is the node-edge incidence matrix of G
  $A_{i,e} = 1$ *if and only if* $i \in e$



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1	1	0	0	0	0	0
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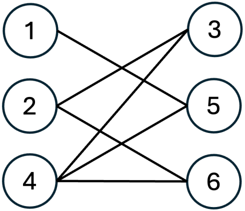

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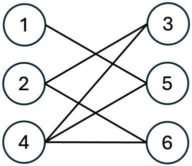
- A **is TU if and only if** G **is bipartite**
  Can partition $\mathcal{N}$ *into* S *and* T *so that every* $e \in E$ *is* $e = (s,t)$ *with* $s \in S, t \in T$

- Bipartite matching problems have integral LP relaxations...

Prove #1: G bipartite implies A is TU



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Proposition (Necessary conditions for $A \in \{-1, 0, 1\}$ to be TU)

1. *$A^\top$ is totally unimodular.*
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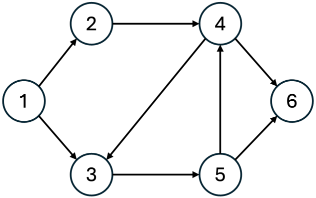
Examples of TU Matrices #2

- $D = (V, A)$ is a **directed graph**
- M is the $V \times A$ incidence matrix of D

$$M_{v,a} = \begin{cases} 1 & \text{if and only if } a = (\cdot, v) \text{ (arc } a \text{ enters node } v) \\ -1 & \text{if and only if } a = (v, \cdot) \text{ (arc } a \text{ leaves node } v) \\ 0 & \text{otherwise.} \end{cases}$$



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- Then M **is TU**
- **Network flow problems** (e.g., Prosche Motors) with integral arc capacities and integral supply/demand have integral LP relaxations

Prove #2 : Incidence Matrix of Directed Graph is TU

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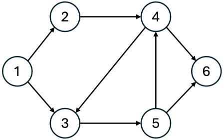

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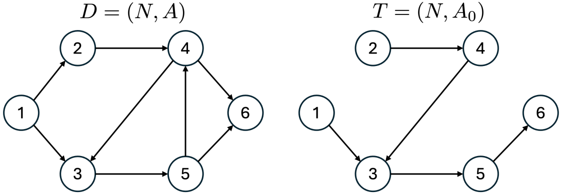

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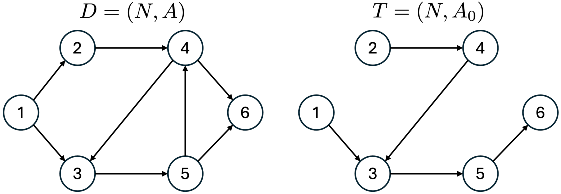

- M is the $A_0 \times A$ matrix defined as follows: for $a = (v, w) \in A$ and $a' \in A_0$,

$$M_{a',a} = \begin{cases} +1 & \text{if the unique } v - w \text{ path in } T \text{ passes through } a' \text{ forwardly} \\ -1 & \text{if the unique } v - w \text{ path in } T \text{ passes through } a' \text{ backwardly} \\ 0 & \text{if the unique } v - w \text{ path in } T \text{ does not pass through } a'. \end{cases}$$

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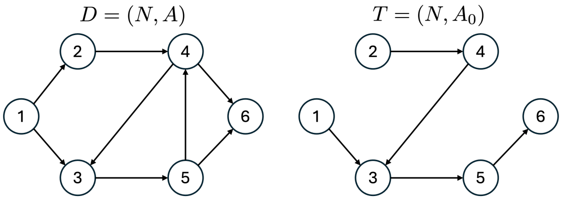

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- Then M **is TU**
- All previous examples were **special cases** of this
- Paul Seymour: **all TU matrices** generated from network matrices and **two** other matrices

Dual Integrality and Submodular Functions

- Alternative conditions based on **LP** duality
- Simple observation: to show that LP relaxation is integral, it suffices to check that the optimal value of any LP with integer cost vector c is an integer

Proposition

P polyhedron with at least one extreme point. Then P is integral **if and only if** *the optimal value $Z_{LP} := \min\{c^T x \mid x \in P\}$ is an integer for all $c \in \mathbb{Z}^n$.*

Why?

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Why?

- To show integrality of P, we **construct an integral dual-optimal solution** (for any $c \in \mathbb{Z}^n$)

- Our discussion here is quite specific
  - broader concepts possible related to Total Dual Integrality
  - if interested, see notes for references

Submodular Functions

Definition

A function $f(S)$ defined on subsets S of a finite set $N = \{1, \ldots, n\}$ is **submodular** if

$$f(S) + f(T) \geq f(S \cap T) + f(S \cup T), \quad \forall S, T \subset N. \tag{1}$$

f is **supermodular** if the reverse inequality holds.

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$$(1) \;\Leftrightarrow f(S) - f(S \cap T) \geq f(S \cup T) - f(T)$$

What is the set difference between arguments on the left? And on the right?

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- **Submodularity: gains when adding something to a smaller set $(S \cap T)$ are larger than when adding it to a larger set (T)**

Submodular Functions - Equivalent Definitions

Proposition

A set function $f : 2^N \to \mathbb{R}$ *is* **submodular if and only if***:*

- **Submodular**: **"diminishing returns"** or **"decreasing differences"**
  - cost: economies of scale/scope
  - profit: substitutability

  Resembles concavity **in economic intuition, but not computationally!** (submodular functions are more like **convex** functions!)
- **Supermodular** is the opposite
- Subsequently, interested in non-negative and **increasing** submodular functions

$$f(S) \leq f(T), \quad \forall S \subset T \subseteq N.$$

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Submodular Functions - Examples

Subsequently, consider a ground set $N = \{1, 2, \ldots, n\}$ and $f : 2^N \to \mathbb{R}$.

- **Linear functions.** For $w \in \mathbb{R}^n$, $f(S) = \sum_{i \in S} w_i$ is both sub- and super-modular.
- **Composition 2.** If $w \geq 0$ and g concave, then $f(S) = g\left(\sum_{i \in S} w_i\right)$ is submodular.
- **Optimal TSP cost on tree graphs.** Consider **undirected tree graph** $G = (N, E)$ with a positive cost for traversing the edges ($c_e \geq 0$ for every edge $e \in E$). For every $S \subseteq N$, define $f(S)$ as the optimal (i.e., smallest) cost for a TSP that goes through all the nodes in S. Then, $f(S)$ is submodular.
- **Network optimization:** consider directed graph with capacities on edges that constrain how much flow can be transported; if $f(S)$ is the maximum flow that can be received at a set of sink nodes S, $f(S)$ is submodular.
- **Operations management and economics:** perishable inventory systems, dual sourcing, inventory control problems with trans-shipment, ...
- **Machine learning and computer vision:** data summarization, distillation, data partitioning / clustering, ...

Main Result

- For a submodular function f, consider the problem:

$$\begin{aligned} \text{maximize } & \sum_{j=1}^{n} r_j \cdot x_j \\ & \sum_{j \in S} x_j \leq f(S), \ \forall S \subseteq N \\ & x \in \mathbb{Z}_+^n. \end{aligned}$$

- T: set of feasible integer solutions
- $P(f)$ the feasible set of the LP relaxation:

$$P(f) = \left\{ x \in \mathbb{R}_+^n \;\middle|\; \sum_{j \in S} x_j \leq f(S), \ \forall S \subset N \right\}$$

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Theorem

If f is submodular, increasing, integer valued, and $f(\emptyset) = 0$, then

$$P(f) = \text{conv}(T).$$

Main Result - Proof

To show: f is submodular, increasing, integer-valued, $f(\emptyset) = 0$, then $P(f) = \mathrm{conv}(T)$.

Proof sketch. Consider the linear relaxation and its dual:

$$\begin{aligned} \text{maximize } & \sum_{j=1}^{n} r_j x_j \\ & \sum_{j \in S} x_j \leq f(S), \quad S \subset N, \\ & x_j \geq 0, \ j \in N \end{aligned}$$

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- Let $S^0 = \emptyset$ and $S^j = \{1, \ldots, j\}$ for $j \in N$.
- Prove that the following x and y are optimal for the primal and dual, respectively.

$$x_j = \begin{cases} f(S^j) - f(S^{j-1}), & 1 \le j \le k, \\ 0, & j > k. \end{cases} \qquad y_S = \begin{cases} r_j - r_{j+1}, & S = S^j, \quad 1 \le j < k, \\ r_k, & S = S^k, \\ 0, & \text{otherwise.} \end{cases}$$

From Discrete to Continuous: The Lovász Extension

- Submodular functions are inherently **discrete**: $f : 2^N \to \mathbb{R}$.
- To connect with convex optimization, we extend f to the **hypercube** $[0, 1]^n$.

From Discrete to Continuous: The Lovász Extension

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- To connect with convex optimization, we extend f to the **hypercube** $[0,1]^n$.
- Given $x \in [0,1]^n$ and a permutation π that sorts coordinates $x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq x_{\pi(n)}$, define:

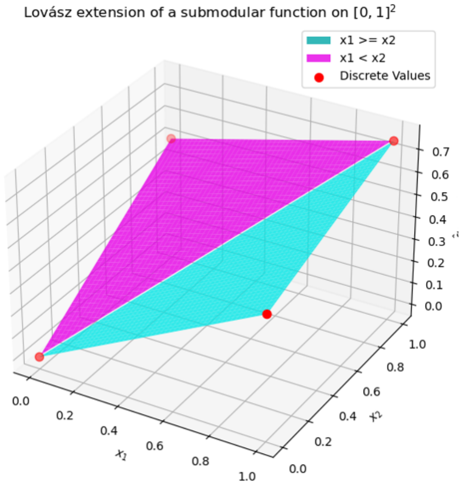
$$\hat{f}(x) = \sum_{k=1}^{n} x_{\pi(k)}\big(f(S_k) - f(S_{k-1})\big), \quad S_k = \{\pi(1), \ldots, \pi(k)\}.$$

- $\hat{f}$ is the **Lovász extension** of f — a piecewise linear interpolation of f's values over the vertices of $[0,1]^n$.

Geometry of the Lovász Extension on $[0,1]^2$

For $N = \{1, 2\}$ with

$$f(S) = \begin{cases} 0, & S = \emptyset, \\ \frac{1}{2}, & S = \{1\} \text{ or } S = \{2\}, \\ \frac{3}{4}, & S = \{1, 2\}. \end{cases}$$



Submodularity & Convexity: The Bridge

Theorem

Key Equivalence (Lovász 1983) A set function $f : 2^N \to \mathbb{R}$ *is* **submodular**

$$\iff \text{its Lovász extension } \hat{f}(x) \text{ is convex on } [0,1]^n.$$

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- Submodularity $\leftrightarrow$ *discrete convexity*.
- Supermodularity $\leftrightarrow$ *discrete concavity*.
- $\hat{f}$ is piecewise linear with gradients corresponding to vertices of the **base polyhedron**

$$B(f) = \{y \in \mathbb{R}^n : y(S) \leq f(S)\ \forall S \subseteq N,\ y(N) = f(N)\}.$$

- Minimizing f over 2^V is equivalent to minimizing $\hat{f}$ over $[0,1]^n$; the minimum is always achieved at a binary vector.

Optimization via the Lovász Extension

Submodular Minimization

- $\min_{S \subseteq N} f(S) = \min_{x \in [0,1]^n} \hat{f}(x)$.
- $\hat{f}$ convex $\Rightarrow$ solvable by convex optimization.
- Algorithms:
  - Iwata–Fleischer–Fujishige (IFF)
  - Schrijver's combinatorial method
  - Subgradient or cutting-plane over $B(f)$

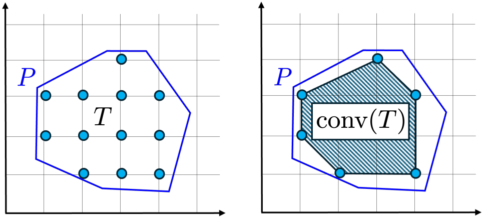
Submodular Maximization

- NP-hard in general (non-convex counterpart).
- Continuous relaxations (multilinear extension) enable approximations.
- Greedy algorithms achieve:

$$1 - \frac{1}{e} \text{ (monotone)}, \quad \frac{1}{2} \text{ (non-monotone)}.$$

Takeaway: The Lovász extension unifies discrete and convex worlds—enabling exact minimization and principled relaxations for maximization.

Improving LP Relaxations With Cuts



- **Recall:** T are feasible points to an IP, $\mathrm{conv}(T)$ is their convex hull
- P is the feasible region of an LP relaxation to the IP
- Typically, the formulation is **not ideal**:

$$\mathrm{conv}(T) \subset P$$

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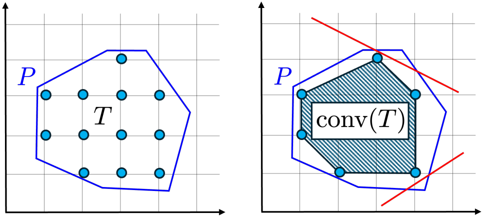

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- Typically, the formulation is **not ideal**:

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- How to **improve it by generating valid cuts**?
  - Linear inequalities **satisfied by** T **and** $\mathrm{conv}(T)$**, but not by** P?

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- **Setup:** A, b, c with rational entries and the IP:

$$\text{minimize}\left\{c^\mathsf{T} x \;:\; Ax = b,\; x \geq 0,\; x \in \mathbb{Z}^n\right\}$$

- If $x^* = [x_B^*; x_N^*]$ be a b.f.s. for the LP relaxation. Then we have:

$$A_B x_B^* + A_N x_N^* = b \quad \Leftrightarrow \quad x_B^* + A_B^{-1} A_N x_N^* = A_B^{-1} b$$

- Consider an equality in which the right-hand-side is **fractional**

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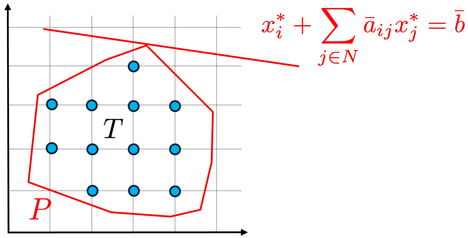

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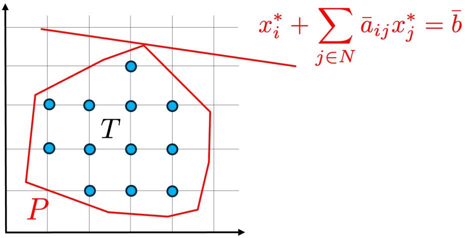

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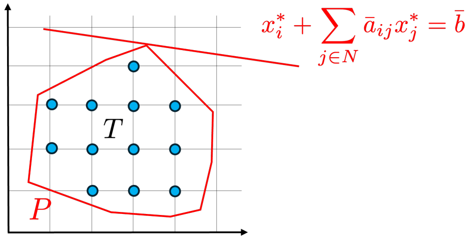

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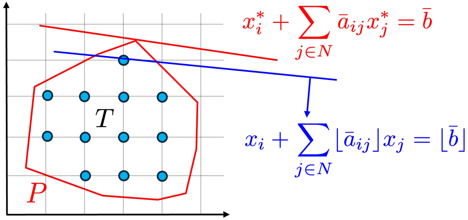

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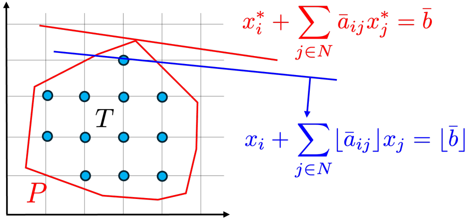
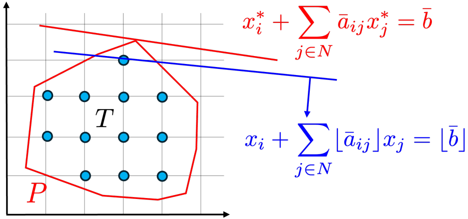


- This inequality is **satisfied by all integer solutions** $x \in T$
- It is **not satisfied by** x^* because $x_i^* = \bar{b}$ is fractional
- **Gomory cut**

Improving LP Relaxations With Cuts

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b} \rfloor, \; \forall x \in T$$



- **Gomory cut**
- Systematically adding all the Gomory cuts lead to first **cutting algorithm** for IP
  1. Solve the linear relaxation and get an optimal solution x^*
  2. If x^* is integer stop
  3. If not, add a cut (i.e., linear inequality that all integer solutions satisfy but that x^* does not satisfy) and go to step 1 again.
- Can show that this is guaranteed to terminate
- *If you're wondering how this works for $Ax \leq b$ or why it terminates, see notes!*

Lift-and-Project

- Balas, Céria and Cornuéjols introduced a new approach
- **Binary** IP, feasible set $x \in P \cap \{0,1\}^n$ where $P := \{x \in \mathbb{R}^n \ : \ Ax \geq b, \, x \geq 0\}$
- **Key idea:** lift linear relaxation polyhedron P to higher dimension where IP formulation is strengthened, and project back

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- **Claims.** (i) Every **binary** $x \in P$ satisfies $x \in P_j$. (ii) $P_j \subseteq P$.
- $\bigcap_{j=1}^n P_j$ is called the **lift-and-project closure**. Clearly, $\bigcap_{j=1}^n P_j \subseteq P$
- Bonami and Minoux : 35 Mixed 0-1 IPs from MIPLIB library, lift-and-project closure reduces integrality gap by **37% on average**

Other Cuts

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- **Knapsack Cover Cuts**: applied for knapsack constraint

$$w \geq 0, w^{\mathsf{T}} x \leq K \;\; \Rightarrow$$

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$$w \geq 0, w^{\mathsf{T}} x \leq K \;\Rightarrow\; \sum_i x_i \leq |C| - 1 \text{ for any } C \;:\; \sum_{i \in C} w_i > K \quad \text{(Cover)}$$

- **Clique Cuts**: used to strengthen $\sum_{i=1}^{n} x_i \leq 1$ when some of the x_i form a **clique**
- **Flow Cover** and **Flow Path Cuts**: specialized cuts for network flow problems
- **Lattice-Free Cuts, Multi-Branch Split Cuts**
- **Comb Inequalities** for TSP
- Solvers like Gurobi have many of these built-in and allow adding custom cuts
- Adding "good" cuts is problem-dependent; requires good understanding of combinatorial structure

Solving IPs

IPs "hard," but many methods devised

- **Exact algorithms**: guaranteed to find optimal solution, but may take exponential number of iterations
  - cutting planes
  - branch and bound
  - branch and cut
  - lift-and-project methods
  - dynamic programming methods
- **Approximation algorithms**: suboptimal solution with a bound on the degree of its suboptimality, in polynomial time
- **Heuristic algorithms**: suboptimal solution, typically no guarantees on its quality; typically run fast
  - local search methods
  - simulated annealing
  - ...

Branch and Bound

Suppose we have **binary** variables **x, y, z** and **minimize an objective**
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- At optimality, get: $x_F=0$, $y_F=0.3$, $z_F=1$
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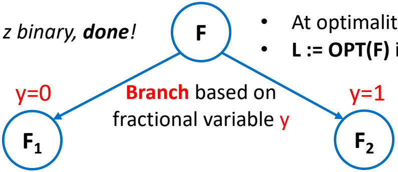

Branch based on fractional variable y

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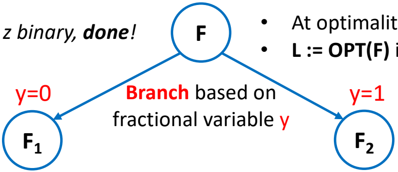

Branch based on fractional variable y

F₁: Solve with y=0, $0 \leq x,z \leq 1$
- Optimal: $x_{F1}=0.5$, $y_{F1}=0$, $z_{F1}=1$
- Optimal value **OPT(F_1)**

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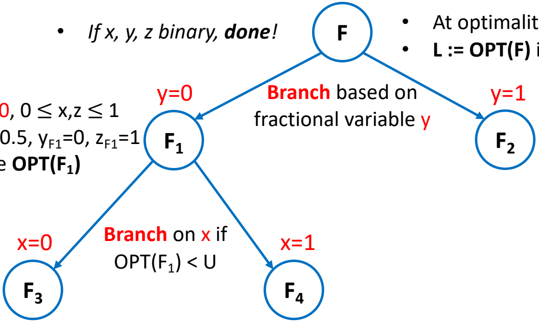

F_1: Solve with y=0, $0 \leq x,z \leq 1$
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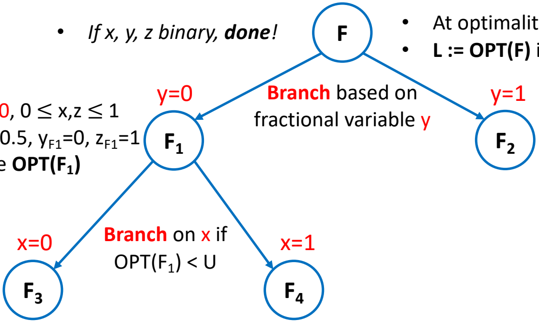

$\mathbf{F_1}$: Solve with y=0, $0 \leq x,z \leq 1$
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$\mathbf{F_3}$: Solve with x=y=0, $0 \leq z \leq 1$
- At optimality: get $z_{F3}=1$
- A **feasible** solution!
- Update **upper bound U := OPT($\mathbf{F_3}$)**
- If **U – L** $\leq$ tolerance, **stop**

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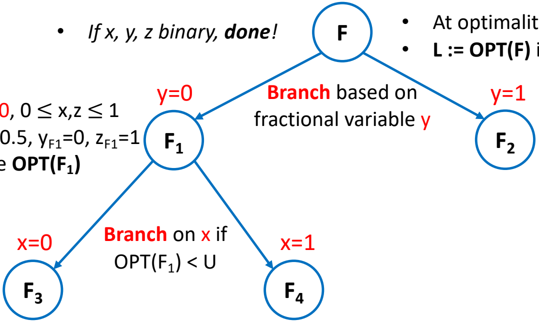

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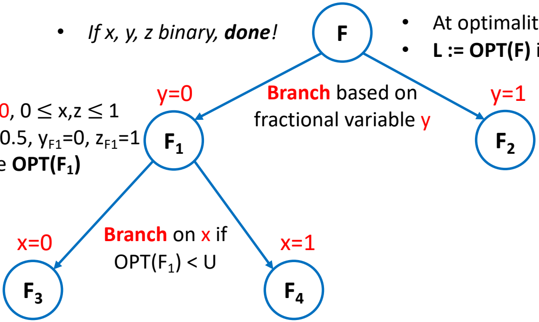

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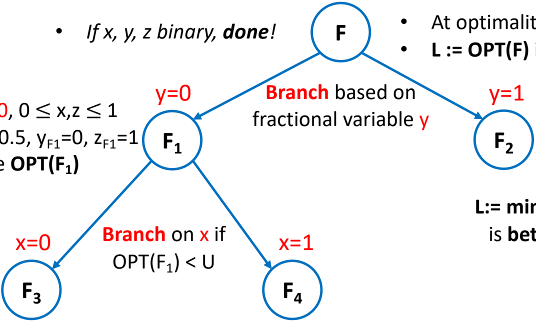

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L:= min($OPT(F_1)$, $OPT(F_2)$) is **better** lower bound!

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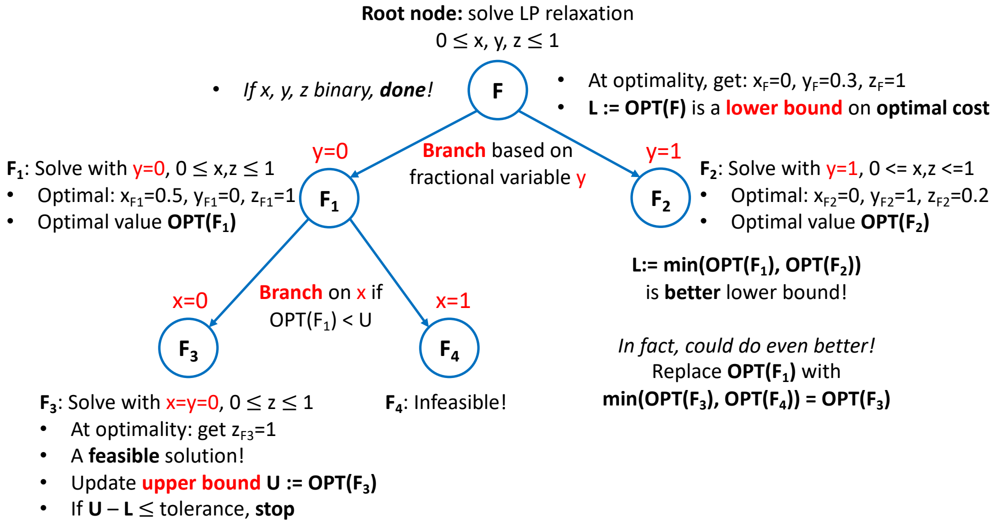

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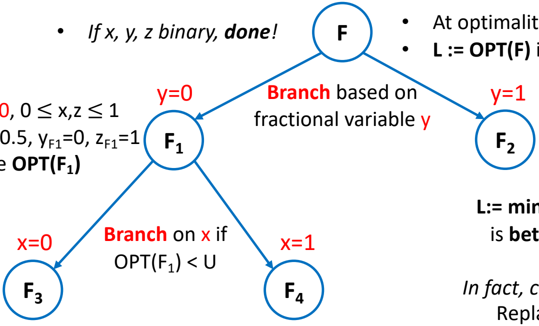

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L:= $\min(OPT(F_1), OPT(F_2))$
is **better** lower bound!

In fact, could do even better!
Replace **$OPT(F_1)$** with
$\min(OPT(F_3), OPT(F_4)) = OPT(F_3)$

F_3: Solve with x=y=0, $0 \leq z \leq 1$
- At optimality: get $z_{F3}=1$
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Trade-off between exploring depth-first vs. breadth-first
- **depth-first** gets to a binary solution "faster"
- **breadth-first** allow improving lower bounds

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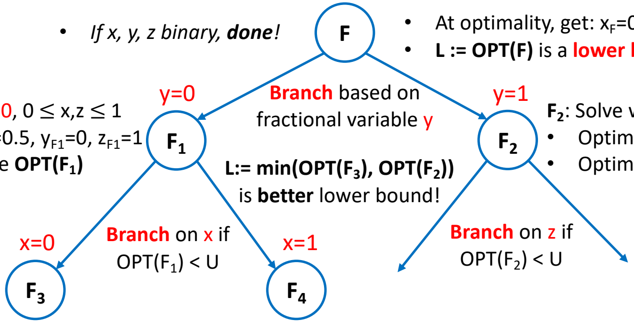

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Branch on x if OPT(F_1) < U

Branch on z if OPT(F_2) < U

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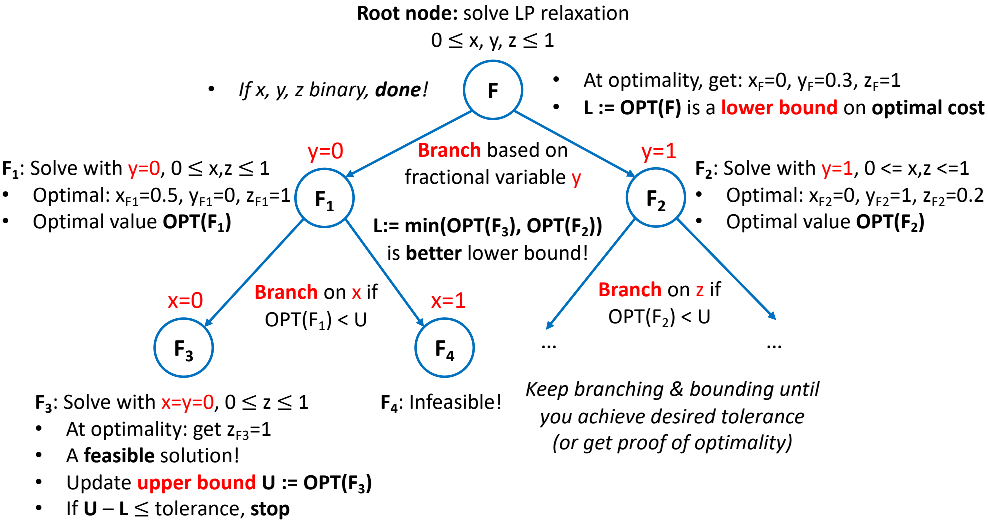

Branch and Bound

- More general formulation: let F be set of feasible solutions to an IP
  1. Maintain upper bound U, lower bound L on problem's objective
  2. Partition F into finite collection of subsets F_i
  3. Choose an unsolved subproblem and solve it; only need a **lower bound** $\ell(F_i)$ on cost:

$$\ell(F_i) \leq \min_{x \in F_i} c^\top x.$$

  4. If $\ell(F_i) \geq U$, no need to explore subproblem F_i further!
  5. Otherwise, partition F_i further and update collection of subproblems/nodes to explore
  6. If we get a feasible solution, update the upper bound U
  7. If $U - L \leq \epsilon$, stop
  8. When solving all children of a given node, can update lower bound at the node

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- Many **choices**:
  1. How to **explore subproblems**: "breadth-first search" vs "depth-first search" vs...
  2. How to **derive lower bound** $\ell(F_i)$: LP relaxation vs. Lagrangean duality
  3. Improve LP relaxations by **adding cuts**: **branch-and-cut** approaches
  4. How to **partition a problem** into subproblems? We used $x_i \leq \lfloor x_i^* \rfloor$ and $x_i \geq \lceil x_i^* \rceil$

Gurobi Output

```
Parameter OutputFlag unchanged
   Value: 1  Min: 0  Max: 1  Default: 1
Gurobi Optimizer version 9.1.2 build v9.1.2rc0 (linux64)
Thread count: 1 physical cores, 2 logical processors, using up to 2 threads
Optimize a model with 55 rows, 105 columns and 310 nonzeros
Model fingerprint: 0x0e3b21e3
Variable types: 5 continuous, 100 integer (100 binary)
Coefficient statistics:
  Matrix range     [5e-02, 1e+00]
  Objective range  [1e+00, 1e+00]
  Bounds range     [1e+00, 1e+00]
  RHS range        [1e+00, 4e+00]
Found heuristic solution: objective -0.0000000
Presolve removed 18 rows and 33 columns
Presolve time: 0.00s
Presolved: 37 rows, 72 columns, 192 nonzeros
Found heuristic solution: objective 1.0190799
Variable types: 0 continuous, 72 integer (68 binary)

Root relaxation: objective 3.139194e+00, 54 iterations, 0.00 seconds

    Nodes    |    Current Node    |     Objective Bounds      |     Work
 Expl Unexpl |  Obj  Depth IntInf | Incumbent    BestBd   Gap | It/Node Time

     0     0    3.13919    0    7    1.01908    3.13919   208%     -    0s
H    0     0                       2.8417259    3.13919  10.5%     -    0s
H    0     0                       3.0648352    3.13919  2.43%     -    0s
H    0     0                       3.0879121    3.13919  1.66%     -    0s
     0     0    3.10586    0    8    3.08791    3.10586  0.58%     -    0s
     0     0     cutoff    0         3.08791    3.08791  0.00%     -    0s

Cutting planes:
  Gomory: 1
  MIR: 1
  GUB cover: 1
  RLT: 1

Explored 1 nodes (57 simplex iterations) in 0.04 seconds
Thread count was 2 (of 2 available processors)

Solution count 5: 3.08791 3.06484 2.84173 ... -0

Optimal solution found (tolerance 1.00e-04)
Best objective 3.087912087912e+00, best bound 3.087912087912e+00, gap 0.0000%

Solved the optimization problem...
```

Available computational resources

Summary of model
constraints, # variables, sparsity, coefficient values

Can we get close with a heuristic?

Can we simplify the problem? (presolve)

Branch & Bound
(current node, bound on objective, gap)

Cutting planes applied

Optimal solution found

Lagrangian Duality in IP

- **Good lower bounds critical for MILPs!**

$$Z_{\mathrm{IP}} := \min \left\{ c^\top x \;:\; Ax \geq b,\; Dx \geq d,\; x \in Z^n \right\}$$

- We get a lower bound from LP relaxation:

$$Z_{\mathrm{LP}} := \min \left\{ c^\top x \;:\; Ax \geq b,\; Dx \geq d \right\} \quad \Rightarrow \quad Z_{\mathrm{LP}} \leq Z_{\mathrm{IP}}$$

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  ... and we are able to **minimize efficiently** $c^\mathsf{T}x$ **over** $\mathcal{X} := \{x \in \mathbb{Z}^n \mid Dx \geq d\}$

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- Let $p \geq 0$ be dual variables (**Lagrange multipliers**) for $Ax \geq b$; form Lagrangean:

$$\mathcal{L}(x, p) := c^\top x + p^\top (b - Ax)$$

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- **Important!** We are **not dualizing** all the constraints!
  - We keep the constraints $x \in \mathcal{X}$ because these are "easy"
  - Similar to LP developments: recall how we kept the constraints $x_i \geq 0$ or $x_i \leq 0$
  - What matters is that we can easily compute $g(p)$ for any $p \geq 0$

Lagrangian Duality in IP

- Because $g(p) \leq Z_{\text{IP}}, \forall p \geq 0$, we can look for **the best lower bound**:

$$Z_D := \max_{p \geq 0} g(p) \tag{2}$$

- This is the **Lagrangean dual** of our problem.
  - $g(p)$ **piece-wise linear, concave**; supergradient available
  - Can compute Z_D using first-order-methods
  - **Weak duality holds:** $Z_D \leq Z_{\text{IP}}$
  - Unlike LP, we do **not** have a strong duality result!

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- Most important result here (recall that $\mathcal{X} := \{x \in \mathbb{Z}^n \mid Dx \geq d\}$)

$$\boxed{Z_D = \min\left\{c^\top x \;:\; Ax \geq b, \quad x \in \mathrm{conv}(\mathcal{X})\right\}.}$$

- Immediate consequence: we get **stronger bounds than from LP relaxation**,

$$Z_{\mathrm{IP}} \leq Z_{\mathrm{D}} \leq Z_{\mathrm{IP}}.$$

- Details, proofs: see notes

Other Methods

- **Dynamic Programming** very powerful
- Can solve in pseudo-polynomial time IPs in **fixed dimension**
- Heuristics can also be powerful
  - Local search
  - Simmulated annealing
  - Genetic algorithms, “ant colony optimization”, etc.