CME 307 / MS&E 311 / OIT 676: Optimization

LP geometry, modeling and solution techniques

Professor Udell

Management Science and Engineering
Stanford

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Course survey

you're interested in:

- modeling real-world problems, from finance and economics to energy systems and trajectory planning
- robustness and modeling under uncertainty
- understanding core optimization concepts like duality
- ...

questions:

- what readings are required?
- what projects are allowed?
- Friday section?
- programming requirements?

Outline

LP standard form

LP inequality form

What kinds of points can be optimal?

Solving LPs

Modeling

Linear programming: standard form

standard form linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

optimal value $p^\star$, solution $x^\star$ (if it exists)

- any x with $Ax = b$ and $x \geq 0$ is called a **feasible point**
- if problem is infeasible, we say $p^\star = \infty$
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A: otherwise infeasible or redundant rows; use gaussian elimination to check and remove

LP example: diet problem

We an planning a backpacking trip, and want to minimize the total weight of the food packed subject to nutritional requirements. We have a list of essential nutrients and how much an active person needs of each. We also know the weight of each food, and how much of each nutrient is in each food.

- x_j servings of food j, $j = 1, \ldots, n$
- c_j weight per serving
- a_{ij} amount of nutrient i in food j
- b_i required amount of nutrient i, $i = 1, \ldots, m$

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- the **positive orthant** $x \geq 0$ is an intersection of halfspaces
- LP is feasible if hyperplane $\{x \mid Ax = b\}$ intersects the positive orthant

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- the intersection of convex sets is convex
- the feasible set $\{x : Ax = b,\ x \geq 0\}$ is convex

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another useful form for LP is **inequality form**

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interpretation: halfspaces

- $a_i^T x \leq b_i$ defines a **halfspace**
- $Ax \leq b$ defines a **polyhedron**: intersection of halfspaces
- LP is feasible if polyhedron $\{x \mid Ax \leq b\}$ is nonempty

LP example: production planning

- x_i units of product i
- c_i cost per unit
- a_{ij} amount of resource j used by product i
- b_j amount of resource j available
- d_i demand for product i

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  $c^T x + f^T z$, $z_i \in \{0, 1\}$, $x_i \leq M z_i$ for M large

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$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array} \quad \rightarrow \quad \begin{array}{ll} \text{minimize} & c^T (x_+ - x_-) \\ \text{subject to} & A(x_+ - x_-) + s = b \\ & s, x_+, x_- \geq 0 \end{array}$$

so both forms have the same expressive power, and feasible sets are polyhedra

Active constraints and variables

active constraints. for constraint set $Ax \le b$, an **active constraint** i at x is one that holds with equality:

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active variables. for nonnegative variable $x \geq 0$, variable i is **active** if $x_i > 0$

example: active slack variables are dual to active constraints

$$\begin{aligned} Ax \leq b \quad &\Longleftrightarrow \quad Ax + s = b,\ s \geq 0 \\ a_i^T x = b_i \quad &\Longleftrightarrow \quad s_i = 0 \\ \text{constraint } i \text{ is active} \quad &\Longleftrightarrow \quad \text{slack variable } s_i \text{ is inactive} \end{aligned}$$

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Q: Does there always exist an extreme solution?

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x is the unique optimum of this problem, so the proof of this statement follows from the previous proof.

Basic feasible solution

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define: $x \in \mathbb{R}^n$ is a **basic feasible solution** (BFS) of (LP) if there is a set $S \subset \{1, \ldots, n\}$ of m columns so that $A_S \in \mathbb{R}^{m \times m}$ is invertible and

$$x_S = A_S^{-1} b, \qquad x_{\bar{S}} = 0, \qquad x \geq 0.$$

- $A_S = \{A_{S_1}, \ldots, A_{S_m}\} \in \mathbb{R}^{m \times m}$ is submatrix of A with columns in S

Basic feasible solution

recall the standard form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \tag{LP}$$

define: $x \in \mathbb{R}^n$ is a **basic feasible solution** (BFS) of (LP) if there is a set $S \subset \{1, \ldots, n\}$ of m columns so that $A_S \in \mathbb{R}^{m \times m}$ is invertible and

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Q: how to find a BFS?
A: choose m linearly independent columns of A and set $x = A_S^{-1} b$; check $x \geq 0$.

Extreme point $\iff$ vertex $\iff$ BFS

fact. consider the feasible set $F = \{x \mid Ax = b,\ x \geq 0\}$ in $\mathbb{R}^n$. the following are equivalent:

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we have already shown that vertex $\Longrightarrow$ extreme point. need to show

- extreme point $\Longrightarrow$ BFS
- BFS $\Longrightarrow$ vertex

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we will show the contrapositive: x is not a BFS $\implies$ x is not an extreme point

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consider $\bar{S} = \{i : x_i^\star > 0\}$, the active set of variables in $\bar{x}$.

- if $A_{\bar{S}}$ were full rank $|\bar{S}|$, we could complete $A_{\bar{S}}$ to an invertible A_S with $\bar{S} \subseteq S$.
- so $A_{\bar{S}}$ has a nontrivial nullspace: there is some $d \in \mathrm{nullspace}(A_{\bar{S}})$, $d_{\bar{S}} \neq 0$.

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extend this vector to $d \in \mathbb{R}^n$ by appending zeros, so $Ad = A_{\bar{S}} d_{\bar{S}} = 0$.
now for $\epsilon \leq \min_{i \in \bar{S}} \bar{x}_i / \max_{i \in \bar{S}} |d_i|$, define $x^+, x^- \in \mathbb{R}^n$ as

$$x^+ = \bar{x} + \epsilon d, \qquad x^- = \bar{x} - \epsilon d.$$

these are feasible:

- $x^+, x^- \geq 0$ by our choice of ϵ,
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so $\bar{x} = \frac{1}{2}x^+ + \frac{1}{2}x^-$ is not extreme in F.

BFS $\implies$ vertex

suppose $x^\star$ is a BFS of F with active set S and A_S invertible. define $c \in \mathbb{R}^n$ as

$$c_i = \begin{cases} 0 & \text{if } i \in S \\ 1 & \text{otherwise} \end{cases}$$

so $c^T x^\star = 0$.

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- $x^\star$ is the only point in F supported on S, as $\mathrm{nullspace}(A_S) = 0$,
- so any other feasible point $x \in F$ has a positive objective value $c^T x > 0$.

hence $x^\star$ is a vertex of F with defining vector c.

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LP inequality form

What kinds of points can be optimal?

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algorithms:

- enumerate all vertices and check
- fourier-motzkin elimination
- simplex method
- ellipsoid method
- interior point methods
- first-order methods
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remarks:

- enumeration and elimination are simple but not practical
- simplex was the first practical algorithm; still used today
- ellipsoid method is the first polynomial-time algorithm; not practical
- interior point methods are polynomial-time and practical
- first-order methods are practical and scale to large problems

Example of Fourier-Motzkin elimination

consider the system of inequalities

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elimination method also shows projection of a polyhedron is a (closed) polyhedron

Enumerate vertices of LP

can generate all extreme points of LP: for each $S \subseteq \{1, \ldots, n\}$ with $|S| = m$,

- $A_S \in \mathbb{R}^{m \times m}$, submatrix of A with columns in S, is invertible
- solve $A_S x_S = b$ for x_S and set $x_{\bar{S}} = 0$
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- evaluate objective $c^T x$

the best BFS is optimal!

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n choose m is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ ("exponentially many")

Simplex algorithm

basic idea: local search on the vertices of the feasible set

- start at BFS x and evaluate objective $c^T x$
- move to a neighboring BFS x' with better objective $c^T x'$
- repeat until no improvement possible

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discuss in groups:

- how to find an initial BFS?
- how to find a neighboring BFS with better objective?
- how to prove optimality?

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- $(x, z) = (x, 0)$ is a BFS of this problem $\iff$ x is a BFS of the original problem

Find a better neighboring BFS

start with BFS x with active set S, $x_S > 0$. (called a **non-degenerate** BFS.)
construct the j**th basic direction** d^j by turning on variable $j \notin S$

$$x^+ \leftarrow x + \theta d^j, \qquad \theta > 0$$

where $d^j_j = 1$ and $d^j_i = 0$ for $i \notin S \cup \{j\}$. need to solve for d^j_S.

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- how does objective change if we move to $x^+ = x + \theta d^j$?

$$c^T x^+ - c^T x = \theta c^T d^j = \theta c_j - \theta c_S^T A_S^{-1} a_j$$

Reduced cost

define **reduced cost** $\bar{c}_j = c_j - c_S^T A_S^{-1} a_j$, $j \notin S$

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fact:

- if $\bar{c} \geq 0$, x is optimal
- if x is optimal and nondegenerate ($x_S > 0$), then $\bar{c} \geq 0$

why might x be degenerate? why might that pose a problem?

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three steps to the proof:

- every feasible direction at x is contained in **cone**$(\{d_j \mid j \notin S\})$

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$$\begin{aligned} p^\star = \min_{x' \in F} c^T x' &\geq \min_{\alpha \geq 0} c^T (x + \sum_{j \notin S} \alpha_j d_j) \\ &= c^T x + \min_{\alpha \geq 0} \sum_{j \notin S} \alpha_j \bar{c}_j = c^T x \end{aligned}$$

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Modeling

Let's do some modeling!

practical solvers for MILP:

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  - power systems https://jump.dev/JuMP.jl/stable/tutorials/applications/power_systems/
  - multicast routing https://colab.research.google.com/drive/1iOn1T1Muh51KaA7mf7UlQOdhSFZhZyry?usp=sharing

Oro Verde case + tutorial

https://github.com/stanford-cme-307/demos/tree/main/gurobipy

Modeling challenges

model the following as standard form LPs:

1. **inequality constraints.** $Ax \leq b$
2. **free variable.** $x \in \mathbb{R}$
3. **absolute value.** constraint $|x| \leq 10$
4. **piecewise linear.** objective $\max(x_1, x_2)$
5. **assignment.** *e.g.*, every class is assigned exactly one classroom
6. **logic.** *e.g.*, class enrollment $\leq$ capacity of assigned room
7. **(big-M).** $Ax \leq b$ if $x \geq 10$
8. **flow.** *e.g.*, the least cost way to ship an item from s to t

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1. **inequality constraints.** $Ax \leq b$
2. **free variable.** $x \in \mathbb{R}$
3. **absolute value.** constraint $|x| \leq 10$
4. **piecewise linear.** objective $\max(x_1, x_2)$
5. **assignment.** *e.g.*, every class is assigned exactly one classroom
6. **logic.** *e.g.*, class enrollment $\leq$ capacity of assigned room
7. **(big-M).** $Ax \leq b$ if $x \geq 10$
8. **flow.** *e.g.*, the least cost way to ship an item from s to t

(see chapter 1 of Bertsimas and Tsitsiklis for more details on 1–6. see https://github.com/stanford-cme-307/demos/blob/main/Mullticast_Routing_Demonstration.ipynb for a detailed treatment of a flow problem.)

Use slack variables to represent inequality constraints

to represent the following problem in standard form,

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

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introduce slack variable $s \in \mathbb{R}^m$: $Ax + s = b$, $s \geq 0 \iff Ax \leq b$

$$\begin{array}{ll} \text{minimize} & c^T x + 0^T s \\ \text{subject to} & Ax + s = b \\ & x, s \geq 0 \end{array}$$

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introduce positive variables x_+, x_- so $x = x_+ - x_-$:

$$\begin{array}{ll} \text{minimize} & c^T x_+ - c^T x_- \\ \text{subject to} & Ax_+ - Ax_- = b \\ & x_+, x_- \geq 0 \end{array}$$

Use epigraph variables to handle absolute value

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$$\begin{array}{ll} \text{minimize} & \|x\|_1 = \sum_{i=1}^n |x_i| \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

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verify these constraints ensure $|x_i| \leq t_i$.

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Q: Why does this work? For what kinds of functions can we use this trick?

Use binary variables to handle assignment

every class is assigned exactly one classroom:
define variable $X_{ij} \in \{0, 1\}$ for each class $i = 1, \ldots, n$ and room $j = 1, \ldots, m$

$$X_{ij} = \begin{cases} 1 & \text{class } i \text{ is assigned to room } j \\ 0 & \text{otherwise} \end{cases}$$

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now solve the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} \\ \text{subject to} & \sum_{j=1}^{m} X_{ij} = 1, \ \forall i \quad \text{(every class assigned one room)} \\ & \sum_{i=1}^{n} X_{ij} \leq 1, \ \forall j \text{(no more than one class per room)} \\ & X_{ij} \in \{0, 1\} \quad \text{(binary variables)} \end{array}$$

where C_{ij} is the cost of assigning class i to room j.

Use binary variables to handle logic

model class enrollment $p_i \leq$ capacity c_j of assigned room:
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where C_{ij} is the cost of assigning class i to room j.

what if we want enrollment p to be a variable, too?

...or use a big-M relaxation!

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where C_{ij} is the cost of assigning class i to room j.