CME 307 / MS&E 311: Optimization

Duality

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Management Science and Engineering
Stanford

May 7, 2023

Announcements

- meet with course staff to discuss project this week or next (see Ed)
- project 1 due this Friday 5/5

Outline

Duality

Lagrange duality

Duality

Definition (Dual space)

The **dual** $\mathcal{X}^*$ of a vector space $\mathcal{X}$ is the set of linear functionals on $\mathcal{X}$.

so if $x \in \mathcal{X}$ and you see someone write

$$w^T x, \qquad \langle w, x \rangle, \qquad \text{or} \qquad w \cdot x$$

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notation: solution to optimization problem $x^\star$ vs dual space $\mathcal{X}^*$

Careful of units!

example 1: suppose $y_i = w^T x_i$ where

$$x_i = \begin{bmatrix} \text{heart rate} \\ \text{blood pressure} \\ \text{age} \end{bmatrix}, \quad \text{with units} \quad \begin{bmatrix} \text{bpm} \\ \text{mmHg} \\ \text{years} \end{bmatrix}$$

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then w has units of

$$\begin{bmatrix} \text{days/bpm} \\ \text{days/mmHg} \\ \text{days/year} \end{bmatrix}$$

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example 2: $f(x) = \sum_{i=1}^{n} \left(1 + \exp\left(\underbrace{y_i w^T x_i}_{\text{input must be a scalar!}}\right)\right)$

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example 3: if $x \in \mathcal{X}$, gradient is a linear function on $\mathcal{X} \implies \nabla f(x_0) \in \mathcal{X}^*$

$$f(x) \approx f(x_0) + \nabla f(x_0)^T (x - x_0),$$

so gradient descent stepsize t has units

$$x^{k+1} = x^k - t\nabla f(x^k)$$

e.g., x (meters m), $\nabla f(x)$ (m^{-1}), and t (m^2)

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- no wonder it's hard to choose the stepsize!
- basic recommendation: standardize your data

Dual of function space

- $f : [0,1] \to \mathbf{R}$ is a function
- $f(x)$ is a linear function of f, for any x:

$$(f+g)(x) = f(x) + g(x), \qquad (cf)(x) = cf(x)$$

- so is any integral:

$$\int_0^1 f(x) d\mu(x)$$

$\implies$ the dual of the space of functions on $[0,1]$ is the space of measures on $[0,1]$

Dual norm

Definition (Dual norm)

The **dual norm** of a norm $\|\cdot\|$ is

$$\|w\|_* = \sup_{\|x\| \leq 1} \langle w, x \rangle$$

equivalently, $\|w\|_* = \sup_x \frac{\langle w,x \rangle}{\|x\|}$

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example: ℓ_1 norm dual is ℓ_∞ norm

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example: for $f : [0,1] \to \mathbf{R}$, if $\|f\| = \sup_{x\in[0,1]} |f(x)|$,

$$\|\mu\|_* \;=\; \sup_{\|f\|\leq 1} \int_0^1 f(x) d\mu(x) = \int_0^1 d|\mu|(x)$$

Self-dual norms

given primal space $\mathcal{X}$

- dual vector is a linear functional $w(x)$ on $x \in \mathcal{X}$
- we should define the dual norm on $\mathcal{X}^*$ as

$$\sup_{x \in \mathcal{X}, \|x\| \leq 1} w(x)$$

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Theorem (Riesz representation)

Suppose $\mathcal{X} = H$ is a Hilbert (inner product) space. For any linear functional $\phi \in \mathcal{X}^*$, there is a unique vector $w \in H$ so that $w(x) = \langle w, x \rangle$ for all $x \in \mathcal{X} = H$. Moreover, $\|w\|_* = \|w\|$.

$\|\cdot\|$ is self-dual $\iff$ $\|\cdot\|$ is induced by an inner product

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$\| \cdot \|$ is self-dual $\iff$ $\| \cdot \|$ is induced by an inner product

example: ℓ_2 norm is self-dual, induced by the inner product

$$\langle w, x \rangle = w^T x$$

Conjugate of linear operator

given $x \in \mathbf{R}^n$, $w \in \mathbf{R}^m$, and $A \in \mathbf{R}^{m \times n}$, conjugate of A is the linear operator A^* defined so that

$$\langle A^* w, x \rangle = \langle w, Ax \rangle$$

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example: $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{m \times n}$ defined by

$$Ax = \begin{bmatrix} x_{i_1} \\ \vdots \\ x_{i_m} \end{bmatrix}$$

then $A^* \in \mathbf{R}^{n \times m}$ satisfies

$$\langle A^* w, x \rangle = \langle w, Ax \rangle = \sum_{j=1}^{m} w_j x_{i_j},$$

so A^* creates a sparse vector from w with

$$(A^* w)_{i_j} = w_j$$

Fenchel dual

Definition (Fenchel dual)

The **Fenchel dual** of a function $f : \mathcal{X} \to \mathbf{R}$ is

$$f^*(w) = \sup_{x \in \mathcal{X}} \langle w, x \rangle - f(x)$$

also called the **conjugate function**. draw picture!
https://remilepriol.github.io/dualityviz/

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example: $f(x) = \|x\|_1$, $x \in \mathbf{R}^n$

$$f^*(w) = \sup_{x \in \mathbf{R}^n} \langle w, x \rangle - \|x\|_1 = \begin{cases} 0 & \|w\|_\infty \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

$\implies$ fenchel dual of ℓ_1 norm is indicator of ℓ_∞ ball

Biconjugate

Definition (Biconjugate)

The **biconjugate** of a function $f : \mathcal{X} \to \mathbf{R}$ is

$$f^{**}(x) = \sup_{w \in \mathcal{X}^*} \langle w, x \rangle - f^*(w)$$

- for convex $f : \mathbf{R} \to \mathbf{R}$, $f^{**} = f$
- for nonconvex f, f^{**} is convex hull of f

$\implies$ biconjugate is a convexification operation

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$\Longrightarrow$ biconjugate is a convexification operation

example: consider $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} 0 & x \in \{-1, 1\} \\ \infty & \text{otherwise} \end{cases}$$

what is f^*? f^{**}?

Outline

Duality

Lagrange duality

Why duality?

- certify optimality
  - turn $\forall$ into $\exists$
  - use dual lower bound to derive stopping conditions
- new algorithms based on the dual
  - solve dual, then recover primal solution

Warmup: Farkas lemma

Theorem (Farkas lemma)

Given $A \in \mathbf{R}^{m \times n}$ *and* $b \in \mathbf{R}^m$*, exactly one of the following is true:*

- *there exists* $x \in \mathbf{R}^n$ *so that* $Ax = b$ *and* $x \geq 0$
- *there exists* $y \in \mathbf{R}^m$ *so that* $A^T y \geq 0$ *and* $\langle b, y \rangle < 0$

$\Longrightarrow$ can efficiently certify infeasibility of a linear program

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proof: suppose we have $x \in \mathbf{R}^n$ so that $Ax = b$ and $x \geq 0$.
then for any $y \in \mathbf{R}^m$,

$$\begin{aligned} 0 &= \langle y, b - Ax \rangle = \langle y, b \rangle - \langle A^T y, x \rangle \\ \langle y, b \rangle &= \langle A^T y, x \rangle \end{aligned}$$

so if $A^T y \geq 0$, then use $x \geq 0$ to conclude $\langle y, b \rangle \geq 0$.

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(opposite direction is similar)

Lagrange duality

primal problem with solution $x^\star \in \mathbf{R}^n$, optimal value $p^\star$:

$$\begin{array}{lll} \text{minimize} & f(x) & \\ \text{subject to} & Ax = b : & \text{dual } y \\ \text{variable} & x \in \mathbf{R}^n & \end{array} \qquad (\mathcal{P})$$

if x is feasible, then $Ax = b$, so $\langle y, Ax - b \rangle = 0$.

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$g(y)$ is called the **dual function**

Lagrange duality

inequality holds for any $y \in \mathbf{R}^m$, so we have proved **weak duality**

$$\begin{aligned} p^\star &\geq g(y) \quad \forall y \in \mathbf{R}^m \\ &\geq \underbrace{\sup_y g(y)}_{\mathcal{D}} =: d^\star \end{aligned} \tag{1}$$

dual optimal value $d^\star \leq p^\star$

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strong duality holds

- for feasible LPs (pf later)
- for convex problems under **constraint qualification** aka **Slater's condition**. feasible region has an **interior point** x so that all inequality constraints hold strictly

strong duality fails if either primal or dual problem is infeasible or unbounded

Lagrange duality with inequality constraints

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this holds for all $y \geq 0$, so we have weak duality

$$p^\star \geq \underbrace{\sup_y g(y)}_{\mathcal{D}} =: d^\star$$

SVM dual

support vector machine: for $x_i \in \mathbf{R}^n$, $y_i \in \{-1, 1\}$, $i = 1, \ldots, m$

$$
\begin{array}{llll}
\text{minimize} & \frac{1}{2}\|w\|^2 + 1^T s & & \\
\text{subject to} & y_i w^T x_i + s_i \geq 1 \quad i = 1, \ldots, m : & \alpha \geq 0 & \text{(SVM)} \\
& s \geq 0 : & \mu \geq 0 &
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verify Slater's condition. strong duality holds!

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$$\mathcal{L}(w, s, \alpha, \mu) = \frac{1}{2}\|w\|^2 + 1^T s - \sum_{i=1}^{m} \alpha_i (y_i w^T x_i + s_i - 1) - \mu^T s$$

- minimize $\mathcal{L}(w, s, \alpha, \mu)$ over w:

$$w = \sum_{i=1}^{m} \alpha_i y_i x_i$$

- minimize $\mathcal{L}(w, s, \alpha, \mu)$ over s $\implies$ $\alpha + \mu = 1$

SVM dual

so simplify:

$$
\begin{aligned}
g(\alpha) &= \inf_{w,s} \mathcal{L}(w, s, \alpha, 1-\alpha) \\
&= \frac{1}{2}\|w\|^2 - w^T \sum_{i=1}^{m} \alpha_i y_i x_i + 1^T \alpha \\
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define $K \in \mathbf{R}^m$ so $K_{ij} = y_i y_j x_i^T x_j$. then

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\|\sum_{i=1}^{m} \alpha_i y_i x_i\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j x_i^T x_j = \alpha^T K \alpha
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dual problem:

$$\begin{array}{ll} \text{maximize} & -\frac{1}{2}\alpha^T K \alpha + 1^T \alpha \\ \text{subject to} & \alpha \geq 0 \end{array} \qquad \text{(SVM-dual)}$$

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define $K \in \mathbf{R}^m$ so $K_{ij} = y_i y_j x_i^T x_j$. then

$$
\|\sum_{i=1}^{m} \alpha_i y_i x_i\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} \alpha_i \alpha_j y_i y_j x_i^T x_j = \alpha^T K \alpha
$$

dual problem:

$$
\begin{array}{ll}
\text{maximize} & -\frac{1}{2}\alpha^T K \alpha + 1^T \alpha \\
\text{subject to} & \alpha \geq 0
\end{array}
\qquad \text{(SVM-dual)}
$$

new solution ideas! proj grad, coord descent (SMO), kernel trick

Generalize Lagrangian duality

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- **nonlinear duality:** replace

$$0 \geq Ax - b \quad \text{with} \quad 0 \geq g(x)$$

(harder to derive explicit form for dual problem)

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- **conic duality:** for cone K, replace

$$b - Ax \geq 0 \quad \text{with} \quad b - Ax \in K$$

define **slack vector** $s = b - Ax \in K$
for weak duality, dual y must satisfy

$$\langle y, s \rangle \geq 0 \quad \forall s \in K$$

Dual cones

this inequality defines the **dual cone** K^*:

Definition (dual cone)

the dual cone K^* of a cone K is the set of vectors y such that

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examples of cones and their duals:

- K acute, K^* obtuse
- $K = \mathbf{R}^m_+$, $K^* = \mathbf{R}^m_+$
- $K = \{x \in \mathbf{R}^n \mid \|x\| \leq x_0\}$, $K^* = \{y \in \mathbf{R}^n \mid \|y\| \leq y_0\}$
- $K = \{X \in \mathbf{S}^n \mid X \succeq 0\}$, $K^* = \{Y \in \mathbf{S}^n \mid Y \succeq 0\}$

inner product $\langle X, Y\rangle = \mathbf{tr}(X^T Y) = \sum_{ij} X_{ij} Y_{ij}$ for $X, Y \in \mathbf{S}^n$

Conic duality

primal problem with solution $x^\star \in \mathbf{R}^n$, optimal value $p^\star$:

$$\begin{array}{ll} \text{minimize} & \langle c, x \rangle \\ \text{subject to} & b - Ax \in K : \quad y \in K^* \\ \text{variable} & x \in \mathbf{R}^n \end{array} \qquad (\mathcal{P})$$

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Dual of the dual

- if $(\mathcal{P})$ is convex, then the dual of (1) is $(\mathcal{P})$
- otherwise, the dual of the dual is the **convexification** of the primal

picture

Strong duality for LPs

primal and dual LP in standard form: (derive!)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \end{array}$$

claim: if primal LP has a bounded feasible solution $x^\star$, then strong duality holds
i.e., dual LP has a bounded feasible solution $y^\star$ and $p^\star = d^\star$

Proof of strong duality for LPs

consider the following system with variables $x' \in \mathbf{R}^n$, $\tau \in \mathbf{R}$

$$Ax' - b\tau = 0, \quad c^T x' = p^\star \tau - 1, \quad (x', \tau) \geq 0$$

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claim: this system has no solution. pf by contradiction:

- if $\tau > 0$, then x'/τ is feasible for LP and $c^T x'/\tau < p^\star$
- if $\tau = 0$, then $x^\star + x'$ is feasible for LP and $c^T(x^\star + x') < p^\star$

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$$Ax + b = 0, \; x \geq 0 \qquad \text{or} \qquad A^T y \geq 0, \quad b^T y < 0$$

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use second system to show y/σ is dual feasible and optimal

Strong duality and complementary slackness

Definition (complementary slackness)

The primal-dual pair x and y are **complementary** if

$$\langle y, b - Ax \rangle = 0$$

They satisfy **strict complementary slackness** if $y_i(b_i - a_i^T x) = 0$ for $i = 1, \ldots, n$.

for conic problem, strong duality $\iff$ complementary slackness

$$\begin{aligned} \langle y, s \rangle &= \langle y, b - Ax \rangle \\ &= \langle y, b \rangle - \langle A^* y, x \rangle \\ &= \langle y, b \rangle - \langle c, x \rangle \end{aligned}$$

KKT conditions

KKT conditions give **necessary** conditions for optimality of convex problem.

Theorem (KKT conditions)

Suppose $x^\star$ and $y^\star$ are primal and dual optimal, respectively. Then

- **stationarity.** *$x^\star$ and $y^\star$ are a min/max saddle point of the Lagrangian*

$$\nabla_x \mathcal{L}(x^\star, y^\star) = 0, \qquad \nabla_y \mathcal{L}(x^\star, y^\star) = 0$$

- **feasibility.** *$x^\star$ is primal feasible; $y^\star$ is dual feasible*
- **complementary slackness.** *$x^\star$ and $y^\star$ are complementary:*

$$\langle y^\star, b - Ax^\star \rangle = 0$$

KKT conditions turn optimization problem into a system of equations

KKT Example

Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & x^2 + y^2 \\ \text{subject to} \quad & x + y \leq 1 : \quad && \lambda \geq 0 \\ & x - y = 0 : \quad && \mu \end{aligned}$$

Lagrangian:

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Lagrangian:

$$\mathcal{L}(x, y, \lambda, \mu) = x^2 + y^2 + \lambda(x + y - 1) + \mu(x - y)$$

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KKT conditions:

1. stationarity: $\nabla L(x, y, \lambda, \mu) = 0$
2. feasibility:
   - primal: $x + y \leq 1$ and $x - y = 0$
   - dual: $\lambda \geq 0$
3. complementary slackness: $\lambda(x + y - 1) = 0$

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Taking the gradient of L wrt x, y, λ, and μ, we get:

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + \lambda + \mu = 0$$
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solve!

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solve! $\rightarrow x^* = 0.5,\ y^* = 0.5,\ \lambda^* = 0,\ \mu^* = 1$