**CME 307: Optimization** CME 307 / MS&E 311 / OIT 676

# Quiz 1: Practice questions

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These questions relate to an LP in standard form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad (\mathcal{P})$$

with $A \in \mathbb{R}^{m \times n}$ of rank $m$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, and $m < n$. Recall that a basic feasible solution (BFS) is a feasible point $x$ for which there is some $S \subseteq [n]$ with $|S| = m$ so that $A_S$ is invertible, $x_S = A_S^{-1} b$, and $x_{\bar{S}} = 0$.

**Exercise.** Rewrite the LP

$$\min\{-2x_1 + x_2\} \quad \text{s.t.} \quad 2x_1 + 3x_2 \leq 5,\ x_1 \in \mathbb{R},\ x_2 \geq 0$$

in standard form by introducing a slack variable and splitting the free variable. Specify the new decision vector and the resulting $(A, b, c)$.

**Exercise.** Let $F = \{(x_1, x_2) \mid x_1 + 2x_2 \leq 4,\ x_1 \geq 0,\ x_2 \geq 0\}$ and $x = (0, 2)$.

(i) List the active constraints at $x$.

(ii) Write the slack form $Ax + s = b$ for the inequality, and give the slack value at $x$.

**Exercise.** Let $A = [A_1\ A_2\ A_3]$ with $A_1 = (1, 0)$, $A_2 = (0, 1)$, $A_3 = (1, 1)$, and $b = (1, 2)$. Decide whether the LP $\{x \geq 0 : Ax = b\}$ is feasible by checking if $b \in \operatorname{cone}(A_1, A_2, A_3)$. If yes, exhibit one nonnegative combination.

**Exercise.** Using the same $A_i$ as above, note that $b = (2, 3)^\top$ can be written as $b = A_1 + 2A_2 + A_3$. Find a representation of $b$ using at most $m = 2$ generators with nonnegative coefficients (as guaranteed by Carathéodory's theorem).

**Exercise.** Take

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x \geq 0.$$

List all BFS: give each basis $S \subset \{1, 2, 3\}$ with $|S| = 2$ that yields $x_S = A_S^{-1} b \geq 0$, and the corresponding $x$.

**Exercise.** Let $x$ be a BFS of $F = \{x \mid Ax = b,\ x \geq 0\}$ with basis $S$. Construct a cost vector $c$ for which $x$ is the unique minimizer of

$$\min\{c^T z \mid z \in F\},$$

thereby proving that every BFS is a vertex.

**Exercise.** Consider

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

At the BFS with basis $S = \{1, 2\}$:

(i) Write the basic direction $d^{(3)}$ for entering variable $j = 3$.

(ii) Compute the reduced cost $\bar{c}_3 = c_3 - c_S^T A_S^{-1} A_3$ and decide if moving along $d^{(3)}$ improves the objective.

(iii) Find the maximal $\theta > 0$ keeping $x + \theta d^{(3)} \geq 0$ and name a valid new basis after the pivot.

**Exercise.** Define the reduced cost $\bar{c}_j$ for $j \notin S$ and prove the reduced cost optimality test: if all reduced costs $\bar{c}_j \geq 0$ for $j \notin S$, then the BFS $x$ is optimal. Explain why the proof fails if the BFS is degenerate.

**Exercise.** Consider the LP

$$\begin{array}{ll} \text{minimize} & x_1 - x_2 \\ \text{subject to} & x_1 + 2x_2 \leq 4 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{array}$$

(a) List all BFS by choosing pairs of active constraints.

(b) For each BFS, compute the objective value.

(c) Identify the optimal BFS.

(d) Verify optimality using reduced costs.

**Exercise.** Consider the feasible set $F = \{x : Ax = b, x \geq 0\}$ of the LP above.

(A) Starting from a nondegenerate BFS $x$ (with $x_S > 0$ for $m$ active variables $S$), what conditions must a vector $d$ satisfy so that $x + \theta d$ is in the feasible set for some $\theta > 0$?

(B) Find a basis for the set of feasible directions $d$. (Hint: suppose $d_j = 1$ for some $j \notin S$, and figure out the other components of $d$.)

**Exercise.** Consider the augmented LP we introduced to help find an initial BFS:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T z \\ \text{subject to} & Ax + Dz = b \\ & x, z \geq 0 \end{array} \qquad (\mathcal{P}')$$

where $D$ is diagonal with $D_{ii} = 1$ if $b_i \geq 0$ and $D_{ii} = -1$ if $b_i < 0$.

(A) What conditions must a BFS $(x, z)$ for this problem satisfy?

(B) Write down a BFS for this problem.

(C) If the original LP is feasible, what can you say about the optimal value of this problem? Give a brief justification.