CME 307: Optimization CME 307 / MS&E 311 / OIT 676

Quiz 3: Practice questions

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Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^4 - 2x^2$.

(i) Find all critical points of f.

(ii) Use the second derivative test to classify each critical point as a local minimizer, local maximizer, or saddle/inflection.

(iii) Determine the global minimizer(s) of f and the global minimum value.

Exercise. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2 - 3x_1$.

(i) Write the first-order optimality condition $\nabla f(x) = 0$.

(ii) Solve for the critical point(s).

(iii) Compute the Hessian and use the second-order condition to classify each critical point. Is any critical point a global minimizer?

Exercise. Let $f(x_1, x_2) = x_1^2 - x_2^2$.

(i) Verify that $\nabla f(0, 0) = 0$.

(ii) Compute the Hessian and determine its eigenvalues at $(0, 0)$. Is it positive (semi)definite?

(iii) Show that $(0, 0)$ is not a local minimizer by exhibiting a direction along which f decreases.

Exercise. Consider the sets in $\mathbb{R}^2$

$$S_1 = \{(x_1, x_2) \mid x_1 + 2x_2 \leq 4, ; x_1 \geq 0, ; x_2 \geq 0\}, \quad S_2 = \{(x_1, x_2) \mid x_1x_2 \geq 1, ; x_1 > 0, ; x_2 > 0\},$$

$$S_3 = \{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}.$$

(i) Prove that S_1 is convex.

(ii) Decide whether S_2 is convex. Justify your answer by the definition (chords) or with a counterexample.

(iii) Prove that S_3 is convex.

Exercise. Define the functions on $\mathbb{R}$:

$$f(x) = \max 2x + 1, -x + 3, \qquad g(x) = e^x, \qquad h(x) = x^3.$$

(i) Prove f is convex using operations that preserve convexity.

(ii) Prove g is convex by an appropriate test.

(iii) Determine whether h is convex on $\mathbb{R}$; justify your answer.

Exercise. Let $f : \mathbb{R}^m \to \mathbb{R}$ be convex and let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$.

(i) Prove that the function $x \mapsto f(Ax + b)$ is convex (use the chord definition).

(ii) Use (i) to show that $x \mapsto \|Ax - b\|_2$ is convex.

(iii) Prove that if f and g are convex, then $f + g$ is convex.

Exercise. (Jensen) Let f be convex and X a random variable.

(i) State Jensen's inequality.

(ii) Let X be uniform on $0, 2$ and $f(t) = t^2$. Compute $f(\mathbb{E}[X])$ and $\mathbb{E}[f(X)]$ and verify Jensen.

(iii) Let X be uniform on $[0, 1]$ and $f(t) = e^t$. Compute $e^{\mathbb{E}[X]}$ and $\mathbb{E}[e^X]$ and verify Jensen.

Exercise. (Subgradients)

(i) Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$. Compute the subdifferential ∂f at $x = 0$.

(ii) For $f(x) = |x|_1$ on $\mathbb{R}^3$, give one subgradient $g \in \partial f(x)$ at $x = (1, 0, -2)$.

(iii) Let $f(x) = \max a_1^\top x + b_1, ; a_2^\top x + b_2$. Describe $\partial f(x)$ when the first affine function is the unique maximizer at x, and when both are maximizers.

Exercise. (Certifying global optimality via first order for convex f)

(i) Prove: If $f : \mathbb{R}^n \to \mathbb{R}$ is convex and differentiable, and $\nabla f(x^\star) = 0$, then $x^\star$ is a global minimizer.

(ii) Let $f(x) = \frac{1}{2}x^\top Qx + q^\top x$ with $Q \succeq 0$. Characterize the set of minimizers and give a condition for uniqueness.

(iii) Apply (ii) to $Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $q = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ to find a minimizer.

Exercise. (Certifying global optimality via subgradient) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = |x| + (x-1)^2$.

(i) Compute $\partial f(x)$ for $x < 0$, for $x = 0$, and for $x > 0$.

(ii) Find $x^\star$ such that $0 \in \partial f(x^\star)$.

(iii) Argue that $x^\star$ is a global minimizer of f.