Practice Problems

Lagrangians, Duals, and Slater's Condition

For each of the following primal optimization problems, address the following issues:
(a) Write the Lagrangian;
(b) Form the dual function $g(\lambda)$ and simplify it as much as you can;
(c) Write the dual problem (including its feasible set);
(d) Is the primal a convex optimization problem?
(e) Does Slater's condition hold for the primal?
(f) Does strong duality hold?

$$(P1) \qquad \min_{x\in\mathbb{R}^n} \ \tfrac{1}{2}\|x\|_2^2 \quad \text{s.t.} \ Ax \geq b \qquad (A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m)$$

$$(P2) \qquad \min_{x\in\mathbb{R}} \ -x^2 \quad \text{s.t.} \ x^2 \leq 1.$$

$$(P3) \qquad \min_{x\in\mathbb{R}^n} \ \sum_{i=1}^n x_i \log x_i \quad \text{s.t.} \ Ax \leq b \quad (x \geq 0 \text{ is included among these}), \quad 1^T x = 1$$

$$(P4) \qquad \min_{x\in\mathbb{R}} \ x \quad \text{s.t.} \ x^2 \geq 0$$

$$(P5) \qquad \min_{x,y\in\mathbb{R}} \ e^x \quad \text{s.t.} \ x^4/y \leq 1$$

$$(P6) \qquad \min_{x\in\mathbb{R}^n} \ \tfrac{1}{2}\|x\|_2^2 \quad \text{s.t.} \ \|Ax-b\|_2 \leq r,$$

$$(P7) \qquad \min_{x\in\mathbb{R}^n} \ \sum_{i=1}^n x_i \log x_i \quad \text{s.t.} \ Ax = b, \quad x \geq 0$$

1 KKT Conditions and Constraint Qualification

For each of the following optimization problems, address the following issues:

(a) Write down the first-order necessary KKT optimality conditions.

(b) Do the Slater constraint qualification conditions hold?

(c) Would these KKT conditions characterize a locally optimal point or the globally optimal point for the problem?

$$(P1) \qquad \min_{x \in \mathbb{R}^2} \ \tfrac{1}{2}(x_1^2 + x_2^2) \quad \text{s.t.} \ x_1 + x_2 = 1, \ x \geq 0$$

$$(P2) \qquad \min_{x \in \mathbb{R}^2} \ x_1^4 + x_2^4 - 3x_1^2 \quad \text{s.t.} \ x_1^2 + x_2^2 \leq 1.$$

$$(P3) \qquad \min_{x \in \mathbb{R}} \ x \quad \text{s.t.} \ x^2 \leq 0.$$

$$(P4) \qquad \min_{x \in \mathbb{R}^2} \ x_1 \quad \text{s.t.} \ x \geq 0, \ x_1 + x_2 \geq 1.$$

$$(P5) \qquad \min_{x \in \mathbb{R}^2} \ x_1 \quad \text{s.t.} \ x_1^2 + x_2^2 = 2.$$

$$(P6) \qquad \min_{x \in \mathbb{R}^2} \ (x_1 - 1)^2 + (x_2 + 2)^2 \quad \text{s.t.} \ x_1 \geq 0, \ x_2 + 3 \geq 0.$$