CME 307: Optimization CME 307 / MS&E 311 / OIT 676

Quiz 7: Practice questions

Fall 2025 *Prof. Udell*

Consider using an interior point method to solve the primal and dual LP

$$\begin{array}{lll} \text{minimize} & c^T x \\ \text{subjecxt to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{lll} \text{maximize} & b^T y \\ \text{subjecxt to} & A^T y + s = c \\ & s \geq 0 \end{array}$$

Exercise. Prove that the complementarity residual $x^T s$ controls the duality gap $c^T x - b^T y$ when x and y are primal feasible and dual feasible, respectively.

Exercise. The IPM stops and returns solution (x, y, s) when the complementarity residual reaches $x^T s \leq \epsilon$. Suppose the primal and dual iterates x and y are feasible. What is the duality gap at this iterate?

Exercise. Write the KKT conditions for the barrier problem with barrier parameter μ, with Lagrangian $L_\mu(x, y, s) = c^T x + y^T(Ax - b) + \mu \sum_{i=1}^n \ln(s)$.

Exercise. Suppose we start our interior point method from a feasible starting point (x, y, s). At each iteration, we move in a direction $(\Delta x, \Delta y, \Delta s)$ satisfying the Newton system

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu_k \mathbf{1} - X^k S^k \mathbf{1} \end{bmatrix}$$

Will a full Newton step to $(x + \Delta x, y + \Delta y, s + \Delta s)$ always be feasible? Why or why not?

Exercise. The Newton system is

$$\begin{bmatrix} A & 0 & 0 \\ -Q & A^T & I \\ S & 0 & X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta s \end{bmatrix} = \begin{bmatrix} b - Ax \\ c - A^\top y - s + Qx \\ \mu_k \mathbf{1} - X^k S^k \mathbf{1} \end{bmatrix}$$

Eliminate s to write the augmented system. What is the benefit of solving the augmented system relative to solving the full Newton system? What are the disadvantages?

Exercise. The Augmented system is

$$\begin{bmatrix} -\Theta^{-1} & A^\top \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \xi_d - X^{-1}\xi_\mu \\ \xi_p \end{bmatrix} = \begin{bmatrix} g \\ \xi_p \end{bmatrix}$$

where $\Theta = XS^{-1}$. Eliminate x to write the normal equations. What are advantages and disadvantages of solving the normal equations relative to the augmented system?

Exercise. Consider the linear program

$$\min_{x \in \mathbb{R}^n} c^\top x \quad \text{s.t.} \quad Ax = b,\ x \geq 0.$$

Define the KKT conditions (primal feasibility, dual feasibility, complementarity) for this LP using variables (x, y, s) with $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$. Then show that along the *central path*—the set of $(x(\mu), y(\mu), s(\mu))$ with $x > 0$, $s > 0$ satisfying

$$Ax = b, \qquad A^\top y + s = c, \qquad XS\mathbf{1} = \mu\mathbf{1},$$

the duality gap satisfies

$$c^\top x - b^\top y \ = \ x^\top s \ = \ n\mu.$$

State any identities you use (e.g., $XS\mathbf{1} = \mu\mathbf{1} \iff s = \mu X^{-1}e$).

Exercise. Numeric central path. Let

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad b = 1, \quad c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mu = \tfrac{1}{2}.$$

Find (x, y, s) with $x > 0, s > 0$ satisfying the perturbed KKT system

$$Ax = b, \qquad A^\top y + s = c, \qquad XS\mathbf{1} = \mu\mathbf{1}.$$

Report x, y, s, and verify that $x^\top s = n\mu$ for $n = 2$.

Exercise. KKT$_\mu$ conditions. Consider the log-barrier Lagrangian for the LP

$$L(x, y; \mu) = c^\top x - y^\top (Ax - b) - \mu \sum_{i=1}^{n} \ln x_i, \qquad x > 0.$$

(a) Compute $\nabla_x L(x, y; \mu)$.

(b) Show that the stationarity condition $\nabla_x L = 0$ can be written as $A^\top y + s = c$ with $s = \mu X^{-1} e$.

(c) Conclude that the log-barrier stationarity plus $Ax = b$ and $x > 0$ is equivalent to the

KKT$_\mu$ system

$$Ax = b, \qquad A^\top y + s = c, \qquad XS\mathbf{1} = \mu\mathbf{1}, \qquad x > 0,\ s > 0.$$

Exercise. Residuals and centrality. Given $A = \begin{bmatrix}1 & 1\end{bmatrix}$, $b = 1$, $c = \begin{bmatrix}1\\1\end{bmatrix}$, $Q = 0$, and a candidate

$$x = \begin{bmatrix}0.6\\0.5\end{bmatrix}, \quad y = -0.2, \quad s = \begin{bmatrix}0.3\\0.4\end{bmatrix},$$

compute:

(a) the primal residual $r_p = Ax - b$,

(b) the dual residual $r_d = A^\top y + s - c$,

(c) the complementarity (centrality) residual $r_c = XS\mathbf{1} - \mu\mathbf{1}$ where $\mu = (x^\top s)/n$.

State n and the numerical value of μ you used.

Exercise. Step lengths to maintain positivity. Let $x = \begin{bmatrix}0.5\\0.5\end{bmatrix}$, $s = \begin{bmatrix}0.4\\0.6\end{bmatrix}$, and Newton directions $\Delta x = \begin{bmatrix}-0.10\\0.05\end{bmatrix}$, $\Delta s = \begin{bmatrix}-0.20\\0.10\end{bmatrix}$.

(a) Compute the largest step sizes that keep positivity:

$$\alpha_{\text{pri,max}} = \max\{\alpha \in (0,1] \mid x + \alpha\Delta x > 0\}, \quad \alpha_{\text{dual,max}} = \max\{\alpha \in (0,1] \mid s + \alpha\Delta s > 0\}.$$

(b) With a safety factor $\tau = 0.95$, set $\alpha_{\text{pri}} = \tau\alpha_{\text{pri,max}}$ and $\alpha_{\text{dual}} = \tau\alpha_{\text{dual,max}}$. Compute the updated $(x^+, s^+) = (x + \alpha_{\text{pri}}\Delta x,\ s + \alpha_{\text{dual}}\Delta s)$ and the new $\mu^+ = (x^{+\top} s^+)/n$.