Practice Problems for Quiz 9 (December 3rd)

December 1, 2025

1. **Comparing two LP relaxations of a simple 0–1 IP.**

   Consider the 0–1 integer program

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & 2x_1 + 2x_2 \ \geq\ 3, \\ & x_1, x_2 \in \{0, 1\}. \end{aligned}$$

   and the following two linear programs:

   Relaxation (L1):

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & 2x_1 + 2x_2 \ \geq\ 3, \\ & 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1. \end{aligned}$$

   Relaxation (L2):

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & 2x_1 + 2x_2 \ \geq\ 3, \\ & x_1 + x_2 \ \geq\ 2, \\ & 0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1. \end{aligned}$$

   (a) Show that both (L1) and (L2) are "relaxations" of the original IP, i.e., their feasible sets contain the feasible set of the original IP.

   (b) Which relaxation is "stronger"?

2. **LP relaxation and a valid inequality.**

   Consider the integer program

$$\begin{aligned} \min\ & x_1 + x_2 \\ \text{s.t.}\ & x_1 + x_2 \ \leq\ 1.5, \\ & x_1, x_2 \in \{0, 1\}. \end{aligned}$$

   (a) List all feasible *integer* solutions (x_1, x_2) of the original IP.

   (b) Write down the natural LP relaxation of this problem.

   (c) Provide a "cut", i.e., a linear inequality that

- is satisfied by *all* feasible integer solutions of the IP, but
- is violated by at least one feasible solution of the LP relaxation.

3. **Proving TU with the given conditions.** Recall the sufficient conditions in Proposition 4.4 that allow testing whether a given matrix A is totally unimodular. *(Note: in the quiz, these conditions would be provided on the slide; you would not need to remember them!)* Using these conditions, prove that the following matrix is TU:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

4. **Checking sub/super-modularity of simple set functions.** Let $N = \{1, 2, 3\}$ and consider the following set functions $f, g, h : 2^N \to \mathbb{R}$:

$$f(S) = |S|, \qquad g(S) = |S|^2, \qquad h(S) = \min\{|S|, 2\}.$$

For each function, decide whether it is submodular or not and provide a brief proof for your answer. You may use any of the following equivalent characterizations of submodularity: A set function $f : 2^N \to \mathbb{R}$ is **submodular if and only if**:

- For any $S, T \subseteq N$,
$$f(S \cap T) + f(S \cup T) \leq f(S) + f(T).$$
- For any $S, T \subseteq N$ such that $S \subseteq T$ and $k \notin T$:
$$f(S \cup \{k\}) - f(S) \geq f(T \cup \{k\}) - f(T).$$
- For any $S \subseteq N$ and any j, k with $j, k \notin S$ and $j \neq k$:
$$f(S \cup \{j\}) - f(S) \geq f(S \cup \{j, k\}) - f(S \cup \{k\}).$$